LACK OF COMPACTNESS IN THE 2D CRITICAL SOBOLEV EMBEDDING, THE GENERAL CASE

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ABSTRACT. This paper is devoted to the description of the lack of compactness of the Sobolev embedding of $H^1(\mathbb{R}^2)$ in the critical Orlicz space $\mathcal{L}(\mathbb{R}^2)$. It turns out that up to cores our result is expressed in terms of the concentration-type examples derived by J. Moser in [38] as in the radial setting investigated in [9]. However, the analysis we used in this work is strikingly different from the one conducted in the radial case which is based on an L^∞ estimate far away from the origin and which is no longer valid in the general framework. Within the general framework of $H^1(\mathbb{R}^2)$, the strategy we adopted to build the profile decomposition in terms of examples by Moser concentrated around cores is based on capacity arguments and relies on an extraction process of mass concentrations. The essential ingredient to extract cores consists in proving by contradiction that if the mass responsible for the lack of compactness of the Sobolev embedding in the Orlicz space is scattered, then the energy used would exceed that of the starting sequence.

CONTENTS

Date: December 14, 2011.

2000 *Mathematics Subject Classification.* ...

Key words and phrases. Sobolev critical exponent, Trudinger-Moser inequality, Orlicz space, lack of compactness, capacity, Schwarz symmetrization.

N. M is partially supported by an NSF Grant DMS-0703145.

1. Introduction

1.1. **Critical $2D$ Sobolev embedding.** It is well known that $H^1(\mathbb{R}^2)$ is continuously embedded in all Lebesgue spaces $L^p(\mathbb{R}^2)$ for $2 \leq p < \infty$, but not in $L^\infty(\mathbb{R}^2)$. On the other hand, it is also known (see for instance [5, 31]) that $H^1(\mathbb{R}^2)$ embeds in $\mathrm{BMO}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, where $BMO(\mathbb{R}^d)$ denotes the space of bounded mean oscillations which is the space of locally integrable functions f such that

$$\|f\|_{BMO} \stackrel{\text{def}}{=} \sup_B \frac{1}{|B|} \int_B |f - f_B|\, dx < \infty \quad \text{with} \quad f_B \stackrel{\text{def}}{=} \frac{1}{|B|} \int_B f\, dx.$$

The above supremum being taken over the set of Euclidean balls B, $|\cdot|$ denoting the Lebesgue measure.

For the sake of geometric problems and the understanding of features of solutions to nonlinear partial differential equations with exponential growth, we investigate in this paper the lack of compactness of Sobolev embedding of $H^1(\mathbb{R}^2)$ in the Orlicz space $\mathcal{L}(\mathbb{R}^2)$ (defined below) which is not comparable to $BMO(\mathbb{R}^2)$ (for a proof of this fact, one can consult [9]). Notice that the lack of compactness of

$$\dot{H}^1(\mathbb{R}^d) \hookrightarrow BMO(\mathbb{R}^d)$$

was characterized in [6] using a wavelet-based profile decomposition.

Note that in higher dimension, are available several works that highlight the role of the study of the lack of compactness in critical Sobolev embedding to the description of bounded energy sequences of solutions of nonlinear partial differential equations. Among others, one can mention [7, 8, 23, 27, 28, 33, 36, 47].

For the convenience of the reader, let us introduce the so-called Orlicz spaces on $\mathbb{R}^d$ and some related basic facts (for a complete presentation and more details, we refer the reader to [39]).

Definition 1.1.
Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex increasing function such that

$$\phi(0) = 0 = \lim_{s \to 0^+} \phi(s), \quad \lim_{s \to \infty} \phi(s) = \infty.$$

We say that a measurable function $u : \mathbb{R}^d \to \mathbb{C}$ belongs to L^ϕ if there exists $\lambda > 0$ such that

$$\int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right) dx < \infty.$$

We denote then

$$\|u\|_{L^\phi} = \inf \left\{ \lambda > 0, \quad \int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right) dx \le 1 \right\}. \tag{1}$$

Remarks 1.2.

• *It is easy to check that* $\|\cdot\|_{L^\phi}$ *is a norm on the* $\mathbb{C}$*-vector space* L^ϕ *which is invariant under translations and oscillations. In particular, we have for any* $u \in L^\phi$

$$\|u\|_{L^\phi} = \| \, |u| \, \|_{L^\phi} \, . \tag{2}$$

• *For* $\phi(s) = s^p$, $1 \le p < \infty$, L^ϕ *is nothing else than the Lebesgue space* L^p.

• *One can easily verify that the number* 1 *in* (1) *may be replaced by any positive constant and that this changes the norm* $\|\cdot\|_{L^\phi}$ *by an equivalent norm.*

In what follows we shall fix $d = 2$, $\phi(s) = \mathrm{e}^{s^2} - 1$ and denote the Orlicz space L^ϕ by $\mathcal{L}$ endowed with the norm $\|\cdot\|_{\mathcal{L}}$ where the number 1 is replaced by the constant κ that will be fixed in identity (4) below. As it is already mentioned, this does not have any impact on the definition of the Orlicz space.

The 2D critical Sobolev embedding in the Orlicz space $\mathcal{L}$ states as follows:

Proposition 1.3.

$$\|u\|_{\mathcal{L}} \le \frac{1}{\sqrt{4\pi}} \|u\|_{H^1}. \tag{3}$$

Let us point out that the embedding (3) derives immediately from the following Trudinger-Moser inequality proved in [40]:

Proposition 1.4.

$$\sup_{\|u\|_{H^1} \le 1} \int_{\mathbb{R}^2} \left(\mathrm{e}^{4\pi |u|^2} - 1 \right) dx := \kappa < \infty, \tag{4}$$

and this is false for $\alpha > 4\pi$.

If we only require that $\|\nabla u\|_{L^2(\mathbb{R}^2)} \le 1$ rather than $\|u\|_{H^1(\mathbb{R}^2)} \le 1$, then the following sharp Trudinger-Moser type inequality holds:

Proposition 1.5. *A constant* C *exists such that*

$$\int_{\mathbb{R}^2} \left(\frac{\mathrm{e}^{4\pi |u|^2} - 1}{1 + |u|^2} \right) dx \le C \|u\|^2_{L^2(\mathbb{R}^2)}, \tag{5}$$

for all u *in* $H^1(\mathbb{R}^2)$ *such that* $\|\nabla u\|_{L^2(\mathbb{R}^2)} \le 1$.

Remarks 1.6.

• *These Trudinger-Moser inequalities established in* [25] *will be of constant use along this paper. Note that a sharp form of Trudinger-Moser inequalities in bounded domain was obtained in* [3] *and a subtle improvement of these inequalities was demonstrated in* [45]. *For further results on the subject, see* [1, 2, 13, 17, 18, 24, 38, 48] *and the references therein.*

• *The injection of* $H^1(\mathbb{R}^2)$ *in* $\mathcal{L}(\mathbb{R}^2)$ *is sharp within the context of Orlicz spaces. However, this embedding can be improved if we allow different function spaces than Orlicz spaces. More precisely*

$$H^1(\mathbb{R}^2) \hookrightarrow BW(\mathbb{R}^2), \tag{6}$$

where the Brezis-Wainger space $BW(\mathbb{R}^2)$ *is defined via*

$$\|u\|_{BW} := \Big(\int_0^1 \Big(\frac{u^\sharp(t)}{\log(\mathrm{e}/t)}\Big)^2 \frac{dt}{t}\Big)^{1/2} + \Big(\int_1^\infty u^\sharp(t)^2\, dt\Big)^{1/2},$$

where $u^\sharp$ *denotes the symmetric decreasing rearrangement of* u*.*

The embedding (6) *is sharper than* (3) *as* $BW(\mathbb{R}^2) \subsetneqq \mathcal{L}(\mathbb{R}^2)$*. It is also optimal with respect to all rearrangement invariant Banach function spaces. For more details on this subject, we refer the reader to* [12, 14, 15, 20, 21, 37]*.*

• *To end this short introduction to Orlicz spaces, let us notice that the Orlicz space* $\mathcal{L}$ *behaves like* L^2 *for functions in* $H^1 \cap L^\infty$ *(see* [9] *for more details).*

1.2. **Development on lack of compactness of Sobolev embedding in the Orlicz space.** The embedding $H^1 \hookrightarrow \mathcal{L}$ is not compact at least for two reasons. The first reason is the lack of compactness at infinity. A typical example is given by $u_k(x) = \varphi(x + x_k)$ where $0 \neq \varphi \in \mathcal{D}$ and $|x_k| \to \infty$. The second reason is of concentration-type derived by P.-L. Lions [34, 35] and illustrated by the following fundamental example f_α defined for $\alpha > 0$ by:

$$f_\alpha(x) = \begin{cases} 0 & \text{if} \quad |x| \geq 1, \\ -\frac{\log|x|}{\sqrt{2\alpha\pi}} & \text{if} \quad \mathrm{e}^{-\alpha} \leq |x| \leq 1, \\ \sqrt{\frac{\alpha}{2\pi}} & \text{if} \quad |x| \leq \mathrm{e}^{-\alpha}. \end{cases}$$

Indeed, it can be seen easily that $f_\alpha \rightharpoonup 0$ in $H^1(\mathbb{R}^2)$ as α tends to either 0 or ∞. However, the lack of compactness of this sequence in the Orlicz space $\mathcal{L}$ occurs only when α goes to infinity and we have by straightforward computations (detailed for instance in [9]):

$$\|f_\alpha\|_{\mathcal{L}} \to \frac{1}{\sqrt{4\pi}} \quad \text{as} \quad \alpha \to \infty \quad \text{and} \quad \|f_\alpha\|_{\mathcal{L}} \to 0 \quad \text{as} \quad \alpha \to 0, \tag{7}$$

and that the essential contribution of $\|f_\alpha\|_{\mathcal{L}}$ when $\alpha \to \infty$ comes from the balls $B(0, \mathrm{e}^{-\alpha})$.

The difference between the behavior of these families in the Orlicz space when $\alpha \to 0$ or $\alpha \to \infty$ comes from the fact that the concentration effect is only displayed by this family when $\alpha \to \infty$.

In fact, in [34, 35], P. -L. Lions highlighted the fact that the defect of compactness of Sobolev embedding in the Orlicz space $\mathcal{L}$, under compactness at infinity,

is due to a concentration phenomena. More precisely, he proved the following result.

Proposition 1.7. *Let (u_n) be a sequence in $H^1(\mathbb{R}^2)$ such that*

$$u_n \rightharpoonup 0 \;\; \textit{in} \;\; H^1, \quad \limsup_{n\to\infty} \|u_n\|_{\mathcal{L}} := A > 0 \;\; \textit{and} \;\; \lim_{R\to\infty} \limsup_{n\to\infty} \|u_n\|_{L^2(|x|>R)} = 0\,.$$

Then, there exists a point $x_0 \in \mathbb{R}^2$ and a constant $c > 0$ such that

$$|\nabla u_n(x)|^2\, dx \rightharpoonup \mu \geq c\,\delta_{x_0}, \;\; \textit{as } n \to \infty, \tag{8}$$

weakly in the sense of measures.

Remarks 1.8.

• *The hypothesis of compactness at infinity*

$$\lim_{R\to\infty} \limsup_{n\to\infty} \|u_n\|_{L^2(|x|>R)} = 0 \tag{9}$$

is necessary to get (8). *For instance,* $u_n(x) = \frac{1}{n}\,\mathrm{e}^{-|\frac{x}{n}|^2}$ *satisfies* $\|u_n\|_{L^2} = \sqrt{\frac{\pi}{2}}$, $\|\nabla u_n\|_{L^2} \to 0$ *and* $\|u_n\|_{\mathcal{L}} \geq \sqrt{\frac{\pi}{2\kappa}}$. *Indeed,*

$$\|u_n\|_{L^2}^2 = \int_{\mathbb{R}^2} \mathrm{e}^{-2|y|^2}\, dy = 2\pi \int_0^\infty r\mathrm{e}^{-2r^2}\, dr = \frac{\pi}{2}$$

and the last assertion follows from the more general estimate

$$\frac{1}{\sqrt{\kappa}}\, \|u\|_{L^2} \leq \|u\|_{\mathcal{L}}\,. \tag{10}$$

In fact by virtue of Rellich's theorem, the compactness at infinity assumption (9) *coupled with the boundedness in H^1 implies the strong convergence of the sequence (u_n) to zero in L^2.*

• *Let us notice that in the case where*

$$|\nabla u_n(x)|^2\, dx \rightharpoonup c\,\delta_{x_0}, \;\; \textit{as } n \to \infty \tag{11}$$

weakly in the sense of measures, we have compactness in the Orlicz space away from the point x_0. More precisely, we have the following lemma:

Lemma 1.9. *Let (u_n) be a bounded sequence in $H^1(\mathbb{R}^2)$ weakly convergent to* 0, *satisfying the hypothesis of compactness at infinity* (9) *and the hypothesis of concentration of the total mass* (11). *Then for any nonnegative reals M and α, we have*

$$\int_{|x-x_0|>M} \left(\mathrm{e}^{|\alpha\, u_n(x)|^2} - 1\right)\, dx \to 0, \quad n \to \infty. \tag{12}$$

Proof. For fixed M, let us consider $\varphi \in \mathcal{C}^\infty(\mathbb{R}^2)$ satisfying $0 \leq \varphi \leq 1$ and such that

$$\begin{cases} \varphi(x) = 0 & \text{if} \quad |x - x_0| \leq \frac{M}{2} \quad \text{and} \\ \varphi(x) = 1 & \text{if} \quad |x - x_0| \geq M. \end{cases}$$

It is obvious that

$$\|\varphi\, u_n\|_{H^1} \stackrel{n\to\infty}{\longrightarrow} 0,$$

which implies for n big enough thanks to Trudinger-Moser estimate (5)

$$\int_{\mathbb{R}^2} \Big(\mathrm{e}^{|\alpha\,\varphi(x)u_n(x)|^2} - 1\Big)\, dx \lesssim \|\varphi\, u_n\|_{L^2}^2 \to 0, \quad \text{as} \quad n\to\infty$$

and ends the proof of the lemma. □

• *In fact in* [34, 35]*, P. -L. Lions relies the defect of compactness of Sobolev embedding in the Orlicz space as well as in Lebesgue spaces to concentration phenomena. Although these results are expressed in the same manner by means of defect measures, they are actually of different nature. Indeed, we shall prove in this article that the lack of compactness of Sobolev embedding in the Orlicz space can be described in terms of an orthogonal asymptotic decomposition whose elements are completely different from the ones involving in the decomposition derived by P. Gérard in* [19] *in the framework of Sobolev embedding in Lebesgue spaces or by S. Jaffard in* [26] *in the more general case of Riesz potential spaces.*
• *Let us mention that the lack of compactness was also studied in* [11] *for a bounded sequence in $H^1_0(D,\mathbb{R}^3)$ of solutions of an elliptic problem, with D the open unit disk of $\mathbb{R}^2$ and in* [43] *and* [42] *for the critical injections of $W^{1,2}(\Omega)$ in Lebesgue space and of $W^{1,p}(\Omega)$ in Lorentz spaces respectively, with Ω a bounded domain of $\mathbb{R}^d$. Similarly, the issue was addressed in* [41] *in an abstract Hilbert space framework and in* [10] *in the Heisenberg group context.*
• *Recently in* [6]*, the wavelet-based profile decomposition introduced by S. Jaffard in* [26] *was revisited in order to treat a larger range of examples of critical embedding of functions spaces $X \hookrightarrow Y$ including Sobolev, Besov, Triebel-Lizorkin, Lorentz, Hölder and BMO spaces and for that purpose, two generic properties on the spaces X and Y was identified to build the profile decomposition in a unified way. (One can consult* [5] *and the references therein for an introduction to spaces listed above).*

In line with the results of P. -L. Lions [34, 35], we investigated in [9] the lack of compactness of Sobolev embedding of $H^1_{rad}(\mathbb{R}^2)$ in the Orlicz space $\mathcal{L}(\mathbb{R}^2)$ following the approach of P. Gérard in [19] and S. Jaffard in [26] which consists in characterizing the lack of compactness of the critical Sobolev embedding in Lebesque spaces in terms of orthogonal profiles. In order to recall our result in a clear way, let us first introduce some objects.

Definition 1.10. *We shall denote by scale any sequence $\underline{\alpha} := (\alpha_n)$ of positive real numbers going to infinity and we shall say that two scales $\underline{\alpha}$ and $\underline{\beta}$ are orthogonal (in short $\underline{\alpha} \perp \underline{\beta}$) if*

$$\Big|\log\left(\beta_n/\alpha_n\right)\Big| \to \infty.$$

We shall designate by profile any function ψ belonging to the set

$$\mathcal{P} := \Big\{\ \psi \in L^2(\mathbb{R}, \mathrm{e}^{-2s}ds); \quad \psi' \in L^2(\mathbb{R}),\ \psi_{|]-\infty,0]} = 0\ \Big\}.$$

Remarks 1.11.
• *The limitation for scales tending to infinity is justified by the behavior of* $\|f_\alpha\|_{\mathcal{L}}$ *stated in* (7).
• *The set* $\mathcal{P}$ *is invariant under positive translations. More precisely, if* $\psi \in \mathcal{P}$ *and* $a \geq 0$ *then* $\psi_a(s) := \psi(s-a)$ *belongs to* $\mathcal{P}$.
• *Let us remark that each profile belongs to the Hölder space* $C^{\frac{1}{2}}(\mathbb{R})$. *In particular, any* $\psi \in \mathcal{P}$ *is continuous.*
• *It will be useful to notice that*

$$\frac{\psi(s)}{\sqrt{s}} \to 0 \quad as \quad s \to 0 \quad and \quad as \quad s \to \infty. \tag{13}$$

Indeed, since $\psi' \in L^2$ *and*

$$\Big|\psi(s)\Big| = \Big|\int_0^s \psi'(\tau)\,d\tau\Big| \leq \sqrt{s}\left(\int_0^s \psi'^2(\tau)\,d\tau\right)^{1/2},$$

we get the first part of the assertion (13). *Similarly, for* $s > A$ *we have*

$$\Big|\frac{\psi(s)}{\sqrt{s}}\Big| \leq \Big|\frac{\psi(s)-\psi(A)}{\sqrt{s}}\Big| + \Big|\frac{\psi(A)}{\sqrt{s}}\Big| \leq \frac{\sqrt{s-A}}{\sqrt{s}}\left(\int_A^s \psi'^2(\tau)\,d\tau\right)^{1/2} + \Big|\frac{\psi(A)}{\sqrt{s}}\Big|,$$

which easily ensures that $\frac{\psi(s)}{\sqrt{s}} \to 0$ *as* s *tends to* ∞. *Now* ψ *being a continuous function, we deduce that the* $\sup\limits_{s>0} \dfrac{|\psi(s)|}{\sqrt{s}}$ *is achieved on* $]0,+\infty[$.
• *Finally, let us observe that for a scale* $\underline{\alpha}$ *and a profile* ψ, *if we denote by*

$$g_{\underline{\alpha},\psi}(x) := \sqrt{\frac{\alpha_n}{2\pi}}\,\psi\Big(\frac{-\log|x|}{\alpha_n}\Big),$$

then for any $\lambda > 0$,

$$g_{\underline{\alpha},\psi} = g_{\lambda\underline{\alpha},\psi_\lambda}, \tag{14}$$

where $\psi_\lambda(t) = \frac{1}{\sqrt{\lambda}}\,\psi(\lambda t)$.

In [9], we established that the lack of compactness of the embedding

$$H^1_{rad} \hookrightarrow \mathcal{L},$$

can be reduced to generalization of the example by Moser. More precisely, we proved that the lack of compactness of this embedding can be described in terms of an asymptotic decomposition as follows:

Theorem 1.12. *Let* (u_n) *be a bounded sequence in* $H^1_{rad}(\mathbb{R}^2)$ *such that*

$$u_n \rightharpoonup 0, \tag{15}$$

$$\limsup_{n\to\infty} \|u_n\|_{\mathcal{L}} = A_0 > 0, \qquad and \tag{16}$$

$$\lim_{R\to\infty} \limsup_{n\to\infty} \|u_n\|_{L^2(|x|>R)} = 0. \tag{17}$$

Then, there exists a sequence $(\underline{\alpha}^{(j)})$ *of pairwise orthogonal scales and a sequence of profiles* $(\psi^{(j)})$ *in* $\mathcal{P}$ *such that, up to a subsequence extraction, we have for all* $\ell \geq 1$,

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\, \psi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x), \quad \limsup_{n\to\infty} \|\mathrm{r}_n^{(\ell)}\|_{\mathcal{L}} \stackrel{\ell\to\infty}{\longrightarrow} 0. \tag{18}$$

Moreover, we have the following orthogonality equality

$$\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|\psi^{(j)\prime}\|_{L^2}^2 + \|\nabla \mathrm{r}_n^{(\ell)}\|_{L^2}^2 + \circ(1), \quad n \to \infty. \tag{19}$$

Remarks 1.13.
• *Note that the example by Moser can be written under the form*

$$f_{\alpha_n}(x) = \sqrt{\frac{\alpha_n}{2\pi}}\, \mathbf{L}\left(\frac{-\log|x|}{\alpha_n}\right),$$

where

$$\mathbf{L}(s) \quad = \quad \begin{cases} 0 & if \quad s \leq 0, \\ s & if \quad 0 \leq s \leq 1, \\ 1 & if \quad s \geq 1. \end{cases}$$

Let us also point out that f_{α_n} *is the minimum energy function which is equal to the value* $\sqrt{\frac{\alpha_n}{2\pi}}$ *on the ball* $B(0, \mathrm{e}^{-\alpha_n})$ *and which vanishes outside the unit ball (see Lemma 3.17 for more details).*
• *The approach that we adopted to prove that result uses in a crucial way the radial setting and particularly the fact we deal with bounded functions far away from the origin thanks to the well known radial estimate (see for instance* [9] *for a sketch of proof).*

$$|u(r)| \leq \frac{C}{\sqrt{r}}\, \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}}. \tag{20}$$

• *It should be emphasized that, contrary to the case of Sobolev embedding in Lebesgue spaces, where the asymptotic decomposition derived by P. Gérard in* [19] *leads to*

$$\|u_n\|_{L^p}^p \to \sum_{j\geq 1} \|\psi^{(j)}\|_{L^p}^p,$$

Theorem 1.12 yields that

$$\|u_n\|_{\mathcal{L}} \to \sup_{j\geq 1}\left(\lim_{n\to\infty} \|g_n^{(j)}\|_{\mathcal{L}}\right), \tag{21}$$

where $g_n^{(j)}(x) = \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\, \psi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right)$.
Let us also notice that

$$\lim_{n\to\infty} \|g_n^{(j)}\|_{\mathcal{L}} = \frac{1}{\sqrt{4\pi}} \max_{s>0} \frac{|\psi^{(j)}(s)|}{\sqrt{s}}. \tag{22}$$

For a detailed proof of (21) *and* (22)*, see Propositions 1.15 and 1.18 in* [9]*. A consequence of* (21) *is that the first profile in Decomposition* (18) *can be chosen such that up to extraction*

$$\limsup_{n\to\infty} \|u_n\|_{\mathcal{L}} = A_0 = \lim_{n\to\infty} \left\| \sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\, \psi^{(1)} \left(\frac{-\log|x|}{\alpha_n^{(1)}} \right) \right\|_{\mathcal{L}} . \tag{23}$$

Otherwise, under the orthogonality assumption of the scales and identity (14)*, we can suppose without loss of generality that for any j*

$$\lim_{n\to\infty} \|g_n^{(j)}\|_{\mathcal{L}} = \frac{1}{\sqrt{4\pi}} \max_{s>0} \frac{|\psi^{(j)}(s)|}{\sqrt{s}} = \frac{1}{\sqrt{4\pi}} |\psi^{(j)}(1)|.$$

• *As a by product of the above remark, it may be noted that while the energy of an elementary concentration $g_n^{(j)}$ remains invariant under translation of the profiles $\psi^{(j)}$, it is not the same for the Orlicz norm. For instance, if we consider the elementary concentration*

$$g_{\alpha_n}(x) = \sqrt{\frac{\alpha_n}{2\pi}}\, \mathbf{L}_a \left(\frac{-\log|x|}{\alpha_n} \right), \quad a \geq 0$$

inferred from example by Moser f_{α_n} by translating the profile, it comes

$$\|g_{\alpha_n}\|_{\mathcal{L}} \to \frac{1}{\sqrt{4\pi(a+1)}},$$

while $\|f_{\alpha_n}\|_{\mathcal{L}} \to \frac{1}{\sqrt{4\pi}}$ as n tends to infinity.

• *Let us observe that each elementary concentration $g_n^{(j)}$ is supported in the unit ball. This is due to the fact that in the radial case, any bounded sequence in $H^1(\mathbb{R}^2)$ is compact away from the origin in the Orlicz space (see Lemma 1.9).*

• *Taking advantage of Proposition 1.3, relations* (18) *and* (19)*, we deduce that in the case where*

$$\|u_n\|_{\mathcal{L}} \sim \frac{1}{\sqrt{4\pi}} \|u_n\|_{H^1}, \tag{24}$$

where the symbol $\sim$ means that the difference goes to zero as n tends to infinity, we have necessary

$$u_n(x) = \sqrt{\frac{\alpha_n}{2\pi}}\, \psi \left(\frac{-\log|x|}{\alpha_n} \right) + \mathrm{r}_n(x), \quad \|\mathrm{r}_n\|_{H^1} \to 0 \quad as \quad n \to \infty,$$

with

$$\psi(s) = \frac{1}{\sqrt{s_0}} \mathbf{L}(\frac{s}{s_0}), \quad for\ some \quad s_0 > 0. \tag{25}$$

Indeed, by virtue of (22)*, equivalence* (24) *leads to* $\|u_n\|_{\mathcal{L}} \sim \frac{1}{\sqrt{4\pi}} \max_{s>0} \dfrac{|\psi(s)|}{\sqrt{s}}$. *Therefore thanks to* (3)*, we get*

$$\|\psi'\|_{L^2} = \max_{s>0} \frac{|\psi(s)|}{\sqrt{s}}.$$

Thus, there exists $s_0 > 0$ *such that*

$$\|\psi'\|_{L^2} = \frac{|\psi(s_0)|}{\sqrt{s_0}}.$$

Since $\frac{|\psi(s_0)|}{\sqrt{s_0}} \leq \left(\int_0^{s_0} |\psi'(t)|^2\,dt\right)^{1/2}$, *we infer that* $\psi' = 0$ *on* $[s_0, +\infty[$ *and then by continuity* $\psi(s) = \sqrt{s_0}$ *for any* $s \geq s_0$. *Finally, the equality case of the Cauchy-Schwarz's inequality*

$$\Big|\psi(s_0)\Big| = \Big|\int_0^{s_0} \psi'(\tau)\,d\tau\Big| = \sqrt{s_0}\left(\int_0^{s_0} \psi'^2(\tau)\,d\tau\right)^{1/2},$$

leads to $\psi(s) = \frac{s}{\sqrt{s_0}}$ *for* $s \leq s_0$ *which ends the proof of claim* (25)*.*

- *Condition* (24) *can be understood as a condition of concentration of the whole mass and plays a crucial role in the qualitative study of solutions of nonlinear wave equations with exponential growth (see* [9] *for more details).*
- *Compared with the decomposition in* [19]*, we see that there is no cores in* (18)*. This is justified by the radial setting.*
- *Note that up to changing the remainders, each profile may be assumed to be in* $\mathcal{D}(]0, +\infty[)$*. This is due to the following lemma that will be useful in the sequel:*

Lemma 1.14. *Let* $\psi \in \mathcal{P}$ *a profile and* $\varepsilon > 0$*. Then, there exists* $\psi_\varepsilon \in \mathcal{D}(]0, +\infty[)$ *such that*

$$\|\psi' - \psi_\varepsilon'\|_{L^2} \leq \varepsilon.$$

Proof. Fix $\psi \in \mathcal{P}$ and $\varepsilon > 0$. Since $\psi' \in L^2$ and $\mathcal{D}$ is dense in L^2, there exists $\chi \in \mathcal{D}$ such that $\|\psi' - \chi\|_{L^2} \leq \frac{\varepsilon}{2}$. Let $\theta_0 \in \mathcal{D}$ satisfying $\int_{\mathbb{R}} \theta_0(s)\,ds = 1$ and set

$$\tilde{\chi} = \chi - \theta^\lambda \int_{\mathbb{R}} \chi(s)\,ds,$$

where $\theta^\lambda(s) = \lambda\,\theta_0(\lambda\, s)$ and λ is a positive constant to be chosen later. Clearly $\tilde{\chi} \in \mathcal{D}$ and $\int_{\mathbb{R}} \tilde{\chi}(s)\,ds = 0$. Hence, there exists a smooth compactly supported function whose derivative is $\tilde{\chi}$. Besides, straightforward computation leads to

$$\|\psi' - \tilde{\chi}\|_{L^2} \leq \frac{\varepsilon}{2} + \|\chi\|_{L^1}\sqrt{\lambda}\,\|\theta_0\|_{L^2},$$

which concludes the proof by choosing $\lambda = \frac{\varepsilon^2}{4\|\chi\|^2_{L^1}\,\|\theta_0\|^2_{L^2}}$. □

- *Finally, let us point out that*

$$|\nabla g_n^{(j)}(x)|^2\,dx \to \|\psi^{(j)'}\|^2_{L^2}\,\delta_0, \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^2). \tag{26}$$

Indeed, straightforward computations give for any smooth compactly supported function φ

$$\begin{aligned}\int |\nabla g_n^{(j)}(x)|^2 \varphi(x)\,dx &= \frac{1}{2\pi\alpha_n^{(j)}}\int_0^1\int_0^{2\pi}\Big|\psi^{(j)\prime}\Big(\frac{-\log r}{\alpha_n^{(j)}}\Big)\Big|^2\,\frac{\varphi(r\cos\theta, r\sin\theta)}{r}\,dr\,d\theta\\ &= \varphi(0)\|\psi^{(j)\prime}\|_{L^2}^2 + I_n^{(j)},\end{aligned}$$

where

$$I_n^{(j)} = \frac{1}{2\pi\alpha_n^{(j)}}\int_0^1\int_0^{2\pi}\Big|\psi^{(j)\prime}\Big(\frac{-\log r}{\alpha_n^{(j)}}\Big)\Big|^2\,\frac{\varphi(r\cos\theta, r\sin\theta)-\varphi(0)}{r}\,dr\,d\theta.$$

Since $\Big|\frac{\varphi(r\cos\theta, r\sin\theta)-\varphi(0)}{r}\Big| \leq \|\nabla\varphi\|_{L^\infty}$, *we deduce that*

$$|I_n^{(j)}| \leq \|\nabla\varphi\|_{L^\infty}\int_0^\infty \Big|\psi^{(j)\prime}(s)\Big|^2\,\mathrm{e}^{-\alpha_n^{(j)}s}\,ds,$$

which ensures the result.

1.3. **Main results.** The heart of this work is to prove that the lack of compactness of the Sobolev embedding $H^1(\mathbb{R}^2) \hookrightarrow \mathcal{L}(\mathbb{R}^2)$ can be reduced up to cores to the example by Moser. Before coming to the statement of the main theorem, let us introduce some definitions as in [9] and [19].

Definition 1.15. *A scale is a sequence* $\underline{\alpha} := (\alpha_n)$ *of positive real numbers going to infinity, a core is a sequence* $\underline{x} := (x_n)$ *of points in* $\mathbb{R}^2$ *and a profile is a function* ψ *belonging to the set*

$$\mathcal{P} := \Big\{\ \psi \in L^2(\mathbb{R}, \mathrm{e}^{-2s}ds);\quad \psi' \in L^2(\mathbb{R}),\ \psi_{|]-\infty,0]} = 0\ \Big\}.$$

Given two scales $\underline{\alpha}$, $\underline{\tilde{\alpha}}$, *two cores* $\underline{x}$, $\underline{\tilde{x}}$ *and two profiles* ψ, $\tilde{\psi}$, *we shall say that the triplets* $(\underline{\alpha}, \underline{x}, \psi)$ *and* $(\underline{\tilde{\alpha}}, \underline{\tilde{x}}, \tilde{\psi})$ *are orthogonal (in short* $(\underline{\alpha}, \underline{x}, \psi) \perp (\underline{\tilde{\alpha}}, \underline{\tilde{x}}, \tilde{\psi})$*) if*

$$\text{either}\qquad \Big|\log\left(\tilde{\alpha}_n/\alpha_n\right)\Big| \longrightarrow \infty\,, \tag{27}$$

or $\tilde{\alpha}_n = \alpha_n$ *and*

$$-\frac{\log|x_n - \tilde{x}_n|}{\alpha_n} \longrightarrow a \geq 0 \text{ with } \psi \text{ or } \tilde{\psi} \text{ null for } s < a\,. \tag{28}$$

Our result states as follows.

Theorem 1.16. *Let* (u_n) *be a bounded sequence in* $H^1(\mathbb{R}^2)$ *such that*

$$u_n \rightharpoonup 0, \tag{29}$$

$$\limsup_{n\to\infty}\|u_n\|_{\mathcal{L}} = A_0 > 0 \qquad \text{and} \tag{30}$$

$$\lim_{R\to\infty}\ \limsup_{n\to\infty}\ \|u_n\|_{\mathcal{L}(|x|>R)} = 0. \tag{31}$$

Then, there exist a sequence of scales $(\underline{\alpha}^{(j)})$, a sequence of cores $(\underline{x}^{(j)})$ and a sequence of profiles $(\psi^{(j)})$ such that the triplets $(\underline{\alpha}^{(j)}, \underline{x}^{(j)}, \psi^{(j)})$ are pairwise orthogonal and, up to a subsequence extraction, we have for all $\ell \geq 1$,

(32)

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\, \psi^{(j)} \left(\frac{-\log|x - x_n^{(j)}|}{\alpha_n^{(j)}} \right) + \mathrm{r}_n^{(\ell)}(x), \quad \limsup_{n\to\infty} \|\mathrm{r}_n^{(\ell)}\|_{\mathcal{L}} \stackrel{\ell\to\infty}{\longrightarrow} 0.$$

Moreover, we have the following orthogonality equality

$$\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|\psi^{(j)\prime}\|_{L^2}^2 + \|\nabla \mathrm{r}_n^{(\ell)}\|_{L^2}^2 + \circ(1), \quad n \to \infty. \tag{33}$$

Remarks 1.17.

• *It may seem surprising that the elements involved in decomposition* (32) *are similar to those that characterize the lack of compactness in the radial case. The idea behind it can be illustrated by the following anisotropic transform of the example by Moser (More general examples will be treated in Appendix 3.1).*

Lemma 1.18. *Let us consider the sequence (u_n) defined by*

$$u_n(x) = f_{\alpha_n}(\lambda_1 x_1, \lambda_2 x_2),$$

where f_{α_n} is the example by Moser and $\lambda_i \in]0, \infty[$, for $i \in \{1, 2\}$. Then

$$u_n \asymp f_{\alpha_n} \quad in \quad \mathcal{L}\,, \tag{34}$$

where the symbol $\asymp$ means that the difference goes to zero in $\mathcal{L}$ as n tends to infinity.

Proof of Lemma 1.18. To go to the proof of Lemma 1.18, let us first consider the case where $\lambda_1 = \lambda_2 = \lambda > 0$, and set

$$v_n(x) = \sqrt{\frac{\alpha_n}{2\pi}}\, \mathbf{L}\Big(\frac{-\log \lambda|x|}{\alpha_n}\Big) - \sqrt{\frac{\alpha_n}{2\pi}}\, \mathbf{L}\Big(\frac{-\log|x|}{\alpha_n}\Big),$$

where $\mathbf{L}$ is the profile by Moser introduced in Remarks 1.13. A straightforward computation leads to

$$\begin{aligned} \|\nabla v_n\|_{L^2}^2 &= 2\|\mathbf{L}'\|_{L^2}^2 - \frac{2}{\alpha_n} \int_0^\infty \mathbf{L}'\left(-\frac{\log \lambda r}{\alpha_n}\right) \mathbf{L}'\left(-\frac{\log r}{\alpha_n}\right) \frac{dr}{r} \\ &:= 2\|\mathbf{L}'\|_{L^2}^2 - 2K_n\,. \end{aligned}$$

The change of variable $s = -\frac{\log r}{\alpha_n}$ yields

$$K_n = \int_{\mathbb{R}} \mathbf{L}'(s)\mathbf{L}'(s - \frac{\log \lambda}{\alpha_n})\, ds\,.$$

But,

$$\Big| K_n - \int_{\mathbb{R}} \mathbf{L}'(s)^2\, ds \Big| \leq \|\mathbf{L}'\|_{L^2} \|\tau_{\frac{\log\lambda}{\alpha_n}} \mathbf{L}' - \mathbf{L}'\|_{L^2}\,.$$

Since $\mathbf{L}' \in L^2(\mathbb{R})$, we have for $\lambda \neq 1$ (the case $\lambda = 1$ being trivial)

$$\|\tau_{\frac{\log \lambda}{\alpha_n}} \mathbf{L}' - \mathbf{L}'\|_{L^2} \to 0 \quad \text{as} \quad n \to \infty,$$

which implies that the sequence (K_n) tends to $\|\mathbf{L}'\|_{L^2}^2$ as n goes to infinity. It follows that $v_n \to 0$ strongly in $H^1(\mathbb{R}^2)$ which leads to the desired conclusion in that case.

Let us now consider the general case where $\lambda_2 < \lambda_1$ and set

$$w_n(x) = u_n(x) - \sqrt{\frac{\alpha_n}{2\pi}}\, \mathbf{L}\Big(\frac{-\log|x|}{\alpha_n}\Big).$$

Using the fact that the profile $\mathbf{L}$ is increasing, we get

$$w_n^1(x) \leq w_n(x) \leq w_n^2(x),$$

where

$$\begin{aligned} w_n^1(x) &= \sqrt{\frac{\alpha_n}{2\pi}}\, \mathbf{L}\Big(\frac{-\log\lambda_1|x|}{\alpha_n}\Big) - \sqrt{\frac{\alpha_n}{2\pi}}\, \mathbf{L}\Big(\frac{-\log|x|}{\alpha_n}\Big), \\ w_n^2(x) &= \sqrt{\frac{\alpha_n}{2\pi}}\, \mathbf{L}\Big(\frac{-\log\lambda_2|x|}{\alpha_n}\Big) - \sqrt{\frac{\alpha_n}{2\pi}}\, \mathbf{L}\Big(\frac{-\log|x|}{\alpha_n}\Big). \end{aligned}$$

We deduce that for all $x \in \mathbb{R}^2$

$$\big|w_n(x)\big| \leq \big|w_n^1(x)\big| + \big|w_n^2(x)\big| \leq \Big|\big|w_n^1(x)\big| + \big|w_n^2(x)\big|\Big|.$$

Taking advantage of invariance under modulus of the Orlicz norm (2) and the monotonicity property (see Lemma 3.1), we conclude the proof of the lemma since the sequences (w_n^1) and (w_n^2) converge strongly to 0 in $H^1(\mathbb{R}^2)$.

□

• *Let us point out that the elementary concentrations*

$$g_n^{(j)}(x) := \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\; \psi^{(j)}\left(\frac{-\log|x - x_n^{(j)}|}{\alpha_n^{(j)}}\right)$$

are completely different from the profiles involved in the characterization of the lack of compactness of the Sobolev embedding investigated in [6], [19] *and* [26]*. In fact, we can prove that for any $0 < a < b$ and any sequence (h_n) of nonnegative real numbers*

$$\int_{a<h_n|\xi|<b} |\widehat{\nabla g_n^{(j)}}(\xi)|^2\, d\xi \to 0, \quad n \to \infty. \tag{35}$$

Actually, the scales $\alpha_n^{(j)}$ do not correspond to scales in the point of view of frequencies as for the profiles describing the lack of compactness of $\dot{H}^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ in [19] *or of $\dot{H}^1(\mathbb{R}^2) \hookrightarrow BMO(\mathbb{R}^2)$ in* [6]*. It was emphasized in* [19] *that the characteristic of being without scales in the sense of* (35) *is measured using the*

Besov norm $\dot B^0_{2,\infty}$ (a precise definition of Besov spaces is available for instance in [5]). We deduce that

$$\|g_n^{(j)}\|_{\dot B^1_{2,\infty}} \to 0, \quad n\to\infty.$$

• *Now in our context, the scales correspond to values taken by the functions $g_n^{(j)}$ in consistent sets of size. More precisely, saying that α_n is a scale for u_n means that $u_n \geq c\sqrt{\alpha_n}$ on a set E_n of Lebesgue measure greater than $\mathrm{e}^{-2\alpha_n}$.*

• *It will be useful later on to observe that if $\|w_n\|_{\mathcal{L}} \stackrel{n\to\infty}{\longrightarrow} 0$, then for any scale α_n, any positive constant C and any ball $B(x_n, \mathrm{e}^{-\alpha_n})$ the sets*

$$F_n := \Big\{x\in\mathbb{R}^2;\ |w_n(x)| \ \geq C\sqrt{\alpha_n}\Big\}$$

satisfy

$$\frac{|F_n\cap B(x_n,\mathrm{e}^{-\alpha_n})|}{|B(x_n,\mathrm{e}^{-\alpha_n})|} \longrightarrow 0, \quad \textit{as} \quad n\to\infty, \tag{36}$$

where $|\cdot|$ denotes the Lebesgue measure. Indeed if $\|w_n\|_{\mathcal{L}} \stackrel{n\to\infty}{\longrightarrow} 0$, then for any $\varepsilon>0$ there exists $N\in\mathbb{N}$ so that for $n\geq N$

$$\int_{\mathbb{R}^2}\left(\mathrm{e}^{\frac{|w_n(x)|^2}{\varepsilon^2}}-1\right)dx \leq \kappa,$$

which implies that

$$|F_n\cap B(x_n,\mathrm{e}^{-\alpha_n})| \lesssim \mathrm{e}^{-\frac{C^2}{\varepsilon^2}\alpha_n}$$

and ensures the result if ε is chosen small enough.

More generally, if for n large $\|w_n\|_{\mathcal{L}}\leq\eta$, then for any scale α_n, any positive constant $C>\sqrt{2}\eta$ and any ball $B(x_n,\mathrm{e}^{-\alpha_n})$

$$\frac{|F_n\cap B(x_n,\mathrm{e}^{-\alpha_n})|}{|B(x_n,\mathrm{e}^{-\alpha_n})|} \longrightarrow 0, \quad \textit{as} \quad n\to\infty.$$

• *As well, if $\|u_n\|_{\mathcal{L}} \stackrel{n\to\infty}{\longrightarrow} A_0$, then for any scale α_n and any measurable set S_n of Lebesgue measure $|S_n|\approx \mathrm{e}^{-c\,\alpha_n}$, the sets*

$$G_n := \Big\{x\in\mathbb{R}^2;\ |u_n(x)| \ \geq M\sqrt{\frac{\alpha_n}{2\pi}}\Big\},$$

with $M>\sqrt{4\pi c}A_0$ check

$$\frac{|G_n\cap S_n|}{|S_n|} \longrightarrow 0, \quad \textit{as} \quad n\to\infty. \tag{37}$$

• *Orthogonality hypothesis means that the interaction between the elementary concentrations is negligible in the energy space. Roughly speaking, condition (28) requires in the case where the cores are not sufficiently distant, namely in the case where $-\frac{\log|x_n-\tilde x_n|}{\alpha_n} \stackrel{n\to\infty}{\longrightarrow} a>0$, the vanishing of one of the profiles for $s<a$. This condition is due to the fact that the parts of the elementary concentrations*

respectively around the cores x_n *and* $\tilde{x}_n$ *resulting from the profiles for the values* $s < a$ *interact. Nevertheless, the parts coming from the profiles for the values* $s > a$ *do not see each other and that is why the vanishing requirement in* (28) *applies only to the zone* $s < a$.

• *The orthogonality assumption* (28) *only concerns the case where the limit* a *is finite. In effect, if the cores are close enough in the sense that* $-\frac{\log|x_n-\tilde{x}_n|}{\alpha_n} \stackrel{n\to\infty}{\longrightarrow} \infty$, *one can add them, up to a remainder term tending to zero in the energy space. More precisely, if we consider the elementary concentration*

$$\sqrt{\frac{\alpha_n}{2\pi}}\,\varphi\left(\frac{-\log|x-x_n|}{\alpha_n}\right),$$

under the assumption $-\frac{\log|x_n|}{\alpha_n} \stackrel{n\to\infty}{\longrightarrow} \infty$, *then by straightforward computations we get*

$$\sqrt{\frac{\alpha_n}{2\pi}}\,\varphi\left(\frac{-\log|x-x_n|}{\alpha_n}\right) \asymp \sqrt{\frac{\alpha_n}{2\pi}}\,\varphi\left(\frac{-\log|x|}{\alpha_n}\right) \quad \text{in} \quad H^1. \tag{38}$$

• *The hypothesis* (31) *of compactness at infinity in the Orlicz space ensures the compactness at infinity in* L^2 *thanks to* (10). *It is necessary to eliminate the lack of compactness of* $H^1(\mathbb{R}^2) \hookrightarrow \mathcal{L}(\mathbb{R}^2)$ *due to translations to infinity mentioned above and illustrated by sequences of type* $u_n(x) = \varphi(x + x_n)$ *where* $0 \neq \varphi \in \mathcal{D}$ *and* $|x_n| \to \infty$. *Such hypothesis allows us, up to a remainder small enough in* $\mathcal{L}$, *to reduce to the case where the mass concentrations are situated in a fixed ball. In particular, for any fixed* j, *we can easily prove that the core* $(\underline{x}^{(j)})$ *is a bounded sequence.*

• *Concerning the behavior of the Orlicz norm, we have the following result:*

Proposition 1.19. *Let* $(\underline{\alpha}^{(j)}, \underline{x}^{(j)}, \psi^{(j)})_{1\le j\le \ell}$ *be a family of triplets of scales, cores and profiles such that the scales are pairwise orthogonal*[1], *and set*

$$g_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\,\psi^{(j)}\left(\frac{-\log|x-x_n^{(j)}|}{\alpha_n^{(j)}}\right) := \sum_{j=1}^{\ell} g_n^{(j)}(x)\,.$$

Then

$$\|g_n\|_{\mathcal{L}} \to \sup_{1\le j\le \ell}\left(\lim_{n\to\infty} \|g_n^{(j)}\|_{\mathcal{L}}\right) = \sup_{1\le j\le \ell}\left(\frac{1}{\sqrt{4\pi}} \max_{s>0} \frac{|\psi^{(j)}(s)|}{\sqrt{s}}\right). \tag{39}$$

• *Let us mention that M. Struwe in* [44] *studied the loss of compactness for the functional*

$$E(u) = \frac{1}{|\Omega|}\int_\Omega \mathrm{e}^{4\pi|u|^2}\,dx,$$

where Ω *is a bounded domain in* $\mathbb{R}^2$. *Also, weak continuity properties of this functional was recently investigated by Adimurthi and K. Tintarev in* [4].

[1]As we will see later in Appendix 3.2, property (39) fails for the same scale and the pairwise orthogonality of the couples $(\underline{x}^{(j)}, \psi^{(j)})$.

1.4. **Layout of the paper.** The paper is organized as follows: in Section 2, we describe the algorithmic construction of the decomposition of a bounded sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(\mathbb{R}^2)$, up to a subsequence extraction, in terms of asymptotically orthogonal profiles concentrated around cores of type $\sqrt{\frac{\alpha_n}{2\pi}}\,\psi(\frac{-\log|x-x_n|}{\alpha_n})$ and prove Theorem 1.16. In the appendix 3, we deal with several complements for the sake of completeness: Appendix 3.1 is dedicated to the study of significant examples, Appendix 3.2 to the proof of Proposition 1.19 specifying the behavior of the decomposition by means of profiles with respect to the Orlicz norm and finally Appendix 3.3 to the collection of all useful known results on rearrangement of functions and the notion of capacity which are used in this text.

Finally, we mention that, C will be used to denote a constant which may vary from line to line. We also use $A \lesssim B$ (respectively $A \gtrsim B$) to denote an estimate of the form $A \leq CB$ (respectively $A \geq CB$) for some absolute constant C and $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$. For simplicity, we shall also still denote by (u_n) any subsequence of (u_n) and designate by $\circ(1)$ any sequence which tends to 0 as n goes to infinity. For two scales (α_n) and (β_n), we shall notice $(\alpha_n) \ll (\beta_n)$ if $\frac{\beta_n}{\alpha_n} \to \infty$.

2. Proof of the main theorem

This section is devoted to the proof of Theorem 1.16. As it is mentioned above, the analysis of the lack of compactness of Sobolev embedding of $H^1(\mathbb{R}^2)$ in the Orlicz space is different from the one conducted in the radial setting [9] where an L^∞ estimate far away from the origin is available. Such L^∞ estimate is obviously not valid in the general case of $H^1(\mathbb{R}^2)$ even far away from a discrete set. To be convenience, it suffices to consider the following bounded sequence in $H^1(\mathbb{R}^2)$

$$u_n(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} f_{\alpha_n}(x - x_k),$$

where (α_n) is a scale, f_{α_n} is the example by Moser and (x_k) is a sequence in $\mathbb{R}^2$ that has accumulation points.

2.1. **The radial case.** Before going into the details of the proof of Theorem 1.16, let us briefly recall the basic idea of the proof of Theorem 1.12 (for more details, one can consult [9]): through a diagonal subsequence extraction, the main step consists to extract a scale (α_n) and a profile ψ such that

$$\|\psi'\|_{L^2} \geq C\, A_0,$$

where C is a universal constant. The extraction of the scale follows from the fact that for any $\varepsilon > 0$

$$\sup_{s\geq 0}\left(\Big|\frac{v_n(s)}{A_0 - \varepsilon}\Big|^2 - s\right) \to \infty, \quad n \to \infty, \tag{40}$$

with $v_n(s) = u_n(\mathrm{e}^{-s})$. Property (40) is proved by contradiction assuming that

$$\sup_{s\geq 0,\, n\in\mathbb{N}} \left(\Big|\frac{v_n(s)}{A_0-\varepsilon}\Big|^2 - s\right) \leq C < \infty,$$

which ensures by virtue of Lebesgue theorem that

$$\int_{|x|<1} \left(\mathrm{e}^{|\frac{u_n(x)}{A_0-\varepsilon}|^2} - 1\right) dx = 2\pi \int_0^\infty \left(\mathrm{e}^{|\frac{v_n(s)}{A_0-\varepsilon}|^2} - 1\right) \mathrm{e}^{-2s}\, ds \to 0, \quad n\to\infty.$$

Furthermore, taking advantage of the radial estimate (20), we deduce that the sequence (u_n) is bounded on the set $\{|x|\geq 1\}$ which implies that

$$\int_{|x|\geq 1} \left(\mathrm{e}^{|\frac{u_n(x)}{A_0-\varepsilon}|^2} - 1\right) dx \leq C\|u_n\|_{L^2}^2 \to 0\,.$$

In conclusion, this leads to

$$\limsup_{n\to\infty} \|u_n\|_{\mathcal{L}} \leq A_0 - \varepsilon,$$

which is in contradiction with hypothesis (16). Fixing $\varepsilon = A_0/2$, a scale (α_n) can be extracted such that

$$\frac{A_0}{2}\sqrt{\alpha_n} \leq |v_n(\alpha_n)| \leq C\sqrt{\alpha_n} + \circ(1).$$

Finally, setting

$$\psi_n(y) = \sqrt{\frac{2\pi}{\alpha_n}}\, v_n(\alpha_n y),$$

one can prove that ψ_n converges simply to a profile ψ. Since $|\psi_n(1)| \geq CA_0$, we obtain

$$CA_0 \leq |\psi(1)| = \Big|\int_0^1 \psi'(\tau)\, d\tau\Big| \leq \|\psi'\|_{L^2(\mathbb{R})},$$

which ends the proof of the main point.

Our approach to characterize the lack of compactness in the general case of the Sobolev embedding

$$H^1(\mathbb{R}^2) \hookrightarrow \mathcal{L}(\mathbb{R}^2)$$

is entirely different from the one conducted in the radial case and uses in a crucial way capacity arguments.

As it was highlighted in Remarks 1.17, the lack of compactness of the Sobolev embedding in the Orlicz space is due to large values of the sequences on sets of significant Lebesgue measure. The main difficulty consists in extracting the cores, which are the points about which enough mass is concentrated. To do so, we will use some capacity arguments and demonstrate by contradiction that the mass responsible for the lack of compactness can not be dispersed.

The proof of Theorem 1.16 is done in four steps. In the first step, using Schawrz symmetrization, we first study u_n^*, the symmetric decreasing rearrangement of u_n (for an introduction to the subject see Appendix 3.3). Since $u_n^* \in H^1_{rad}(\mathbb{R}^2)$ and satisfies assumptions of Theorem 1.12, it can be written as an orthogonal asymptotic decomposition by means of elementary concentrations similar to examples of Moser, namely of type $g_n^{(j)}(x) = \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\,\varphi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(1)}}\right)$. This actually yields the scales of u_n too. Then, in the second step, taking advantage of (23), we reduce ourselves to one scale and extract the first core $(x_n^{(1)})$ and the first profile $\psi^{(1)}$ which leads to the extraction of the first element $\sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\,\psi^{(1)}\left(\frac{-\log|x-x_n^{(1)}|}{\alpha_n^{(j)}}\right)$. This step constitutes the heart of the proof and relies on capacity arguments: the basic idea is to show that the mass does not scatter and is mainly concentrated around some points that will constitute the cores. In the third step, we study the remainder term. If the limit of its Orlicz norm is null we stop the process. If not, we prove that this remainder term satisfies the same properties as the sequence we start with which allows us to extract a second elementary concentration concentrated around a second core $(x_n^{(2)})$. Thereafter, we establish the property of orthogonality between the first two elementary concentrations and finally we prove that this process converges.

2.2. **Extraction of the scales.** Using Schawrz symmetrization, we first study u_n^*, the symmetric decreasing rearrangement of u_n, (see Appendix 3.3 for all details). Since $u_n^* \in H^1_{rad}(\mathbb{R}^2)$ and satisfies by virtue of Proposition 3.11 the assumptions of Theorem 1.12, there exists a sequence $(\underline{\alpha}^{(j)})$ of pairwise orthogonal scales and a sequence of profiles $(\varphi^{(j)})$ in $\mathcal{P}$ such that, up to a subsequence extraction, we have for all $\ell \geq 1$,

$$u_n^*(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\,\varphi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x), \quad \limsup_{n\to\infty}\ \|\mathrm{r}_n^{(\ell)}\|_{\mathcal{L}} \overset{\ell\to\infty}{\longrightarrow} 0.$$

Since u_n^* is nonnegative, it is clear under the proof sketch presented in Section 2.1 that the profiles $\varphi^{(j)}$ are nonnegative. Moreover, thanks to (23) we can suppose that

$$A_0 = \lim_{n\to\infty}\left\|\sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\,\varphi^{(1)}\left(\frac{-\log|x|}{\alpha_n^{(1)}}\right)\right\|_{\mathcal{L}}$$

with $\displaystyle\max_{s>0}\frac{\varphi^{(1)}(s)}{\sqrt{s}} = \varphi^{(1)}(1)$. It follows clearly that $\varphi^{(1)}(1) = \sqrt{4\pi}\,A_0$ and

$$\varphi^{(1)}(s) \leq \sqrt{s}\sqrt{4\pi}\,A_0,\ \forall s \geq 0. \tag{41}$$

Consequently, for any $0 < \varepsilon < 1$ and $s \leq 1-\varepsilon$, we get that

$$\varphi^{(1)}(s) \leq (1-\frac{\varepsilon}{2})\sqrt{4\pi}\,A_0. \tag{42}$$

Now, let us consider $0<\varepsilon_0<\frac{1}{2}$ to be fixed later on and denote by E_n^* the set

$$E_n^* := \Big\{ x\in\mathbb{R}^2;\ g_n^{(1)}(x)\ \geq \sqrt{2\alpha_n^{(1)}}\,\Big(1-\frac{\varepsilon_0}{10}\Big)\,A_0\Big\},$$

where $g_n^{(1)}(x)=\sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\,\varphi^{(1)}\Big(\frac{-\log|x|}{\alpha_n^{(1)}}\Big)$. Then, we have the following useful result:

Lemma 2.1. *There exists an integer N_0 such that for $n\geq N_0$*

$$|E_n^*|\geq \mathrm{e}^{-2\alpha_n^{(1)}}, \tag{43}$$

where $|\cdot|$ denotes the Lebesgue measure.

Proof. By definition, saying that x belongs to E_n^* is equivalent to saying that

$$\varphi^{(1)}\Big(\frac{-\log|x|}{\alpha_n^{(1)}}\Big)\geq\Big(1-\frac{\varepsilon_0}{10}\Big)\sqrt{4\pi}\,A_0.$$

Since $\varphi^{(1)}$ is continuous, there exists $\eta>0$ such that

$$|s-1|\leq\eta\Rightarrow\Big|\varphi^{(1)}(s)-\varphi^{(1)}(1)\Big|\leq\frac{\varepsilon_0}{10}\sqrt{4\pi}\,A_0.$$

Knowing that $\varphi^{(1)}(1)=\sqrt{4\pi}\,A_0$, we deduce that $\varphi^{(1)}(s)\geq\Big(1-\frac{\varepsilon_0}{10}\Big)\sqrt{4\pi}\,A_0$ provided that $|s-1|\leq\eta$. In other words, for $\mathrm{e}^{-\alpha_n^{(1)}(1+\eta)}\leq|x|\leq\mathrm{e}^{-\alpha_n^{(1)}(1-\eta)}$, we have

$$g_n^{(1)}(x)\geq\sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\,\Big(1-\frac{\varepsilon_0}{10}\Big)\sqrt{4\pi}\,A_0.$$

This achieves the proof of the lemma. □

Remark 2.2. *It is clear that we have shown above that there is $\eta>0$ such that for n big enough $|E_n^*|\gtrsim\mathrm{e}^{-2\alpha_n^{(1)}(1-\eta)}$. In fact in light of estimate (42), we have $|E_n^*|\lesssim\mathrm{e}^{-2\alpha_n^{(1)}(1-\frac{\varepsilon_0}{5})}$.*

2.3. **Reduction to one scale.** Our aim now is to reduce to one scale. For this purpose, we introduce, for any $0<a<M$, the odd cut-off function Θ_a^M as follows

$$\Theta_a^M(s)\;=\;\begin{cases}0 & \text{if}\quad 0\leq s\leq a/2,\\ 2s-a & \text{if}\quad a/2\leq s\leq a,\\ s & \text{if}\quad a\leq s\leq M,\\ M & \text{if}\quad s\geq M.\end{cases}$$

The following result concerning the truncation of u_n at the scale $\alpha_n^{(1)}$ will be crucial in the sequel:

Proposition 2.3. *Let (u_n) be a sequence in $H^1(\mathbb{R}^2)$ satisfying the assumptions of Theorem 1.16, and define for fixed $0<a<M$,*

$$\widetilde{u}_{n,a}^M := \sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\,\Theta_a^M\left(\sqrt{\frac{2\pi}{\alpha_n^{(1)}}}\,u_n\right).$$

Then for any $\varepsilon>0$, there exist $N\in\mathbb{N}$ and $0<a<M$ such that

$$\left(\widetilde{u}_{n,a}^M\right)^*(x) = \sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\,\varphi^{(1)}\left(\frac{-\log|x|}{\alpha_n^{(1)}}\right) + \widetilde{r}_n(x),$$

with $\|\widetilde{\mathrm{r}}_{\mathrm{n}}\|_{\mathcal{L}} \le \varepsilon$ for any $n\ge N$.

Proof. In light of Proposition 3.11, u_n^* satisfies hypothesis of Theorem 1.12 which allows us to find $\ell\ge 1$ such that

$$\begin{aligned} u_n^*(x) &= \sum_{j=1}^{\ell}\sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\,\varphi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x), \quad \limsup_{n\to\infty}\ \|\mathrm{r}_n^{(\ell)}\|_{\mathcal{L}} \le \frac{\varepsilon}{2}\\ &= \sum_{j=1}^{\ell} g_n^{(j)}(x) + \mathrm{r}_n^{(\ell)}(x), \end{aligned}$$

where $(\underline{\alpha}^{(j)})$ is a sequence of pairwise orthogonal scales and $(\varphi^{(j)})$ a sequence of profiles in $\mathcal{P}$.

Since the cut-off function Θ_a^M is non-decreasing, it comes in view of (118)

$$\begin{aligned} \left(\widetilde{u}_{n,a}^M\right)^*(x) &= \sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\,\Theta_a^M\left(\sqrt{\frac{2\pi}{\alpha_n^{(1)}}}\,u_n^*(x)\right)\\ &= \sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\,\Theta_a^M\left(\sqrt{\frac{2\pi}{\alpha_n^{(1)}}}\left(\sum_{j=1}^{\ell} g_n^{(j)}(x) + \mathrm{r}_n^{(\ell)}(x)\right)\right). \end{aligned}$$

We are then reduced to the proof of the following lemma.

Lemma 2.4. *Let $(\underline{\alpha}^{(j)})_{1\le j\le \ell}$ be a family of pairwise orthogonal scales, $(\varphi^{(j)})_{1\le j\le\ell}$ a family of profiles, and set*

$$v_n(x) = \sum_{j=1}^{\ell}\sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\,\varphi^{(j)}\left(-\frac{\log|x|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x),$$

where $\limsup_{n\to\infty}\ \|\mathrm{r}_n^{(\ell)}\|_{\mathcal{L}} \le \frac{\varepsilon}{2}$. Then, for any $1\le k\le\ell$, we have (as $n\to\infty$),

(44)
$$\limsup_{n\to\infty}\left\|\sqrt{\frac{\alpha_n^{(k)}}{2\pi}}\,\Theta_a^M\left(\sqrt{\frac{2\pi}{\alpha_n^{(k)}}}\,v_n(x)\right) - \sqrt{\frac{\alpha_n^{(k)}}{2\pi}}\,\Theta_a^M\left(\varphi^{(k)}\left(-\frac{\log|x|}{\alpha_n^{(k)}}\right)\right)\right\|_{\mathcal{L}} \le \frac{\varepsilon}{2}.$$

In particular, the profile associated to $\sqrt{\frac{\alpha_n^{(k)}}{2\pi}}\,\Theta_a^M\left(\sqrt{\frac{2\pi}{\alpha_n^{(k)}}}\,u_n^*\right)$ *is* $\Theta_a^M\circ\varphi^{(k)}$.

Proof. For simplicity, we assume that $\ell=2$ and write

$$v_n(x)=\sqrt{\frac{\alpha_n}{2\pi}}\,\varphi\left(-\frac{\log|x|}{\alpha_n}\right)+\sqrt{\frac{\beta_n}{2\pi}}\,\psi\left(-\frac{\log|x|}{\beta_n}\right)+\mathrm{r}_n(x)\,,$$

where $(\alpha_n)\perp(\beta_n)$ are two scales, φ,ψ are two profiles, and $\|\mathrm{r}_n\|_{\mathcal{L}}\to 0$ as $n\to\infty$.

Let $\varepsilon>0$ to be chosen later. It suffices to prove that, for some $\lambda=\lambda(\varepsilon)\to 0$ as ε goes to zero, we have (for n big enough)

$$\int_{\mathbb{R}^2}\left(\mathrm{e}^{|\frac{g_n(x)}{\lambda}|^2}-1\right)dx\leq\kappa,\tag{45}$$

where[2]

$$g_n(x)=\sqrt{\frac{\alpha_n}{2\pi}}\left[\Theta\left(\varphi\left(-\frac{\log|x|}{\alpha_n}\right)+\sqrt{\frac{\beta_n}{\alpha_n}}\,\psi\left(-\frac{\log|x|}{\beta_n}\right)+\sqrt{\frac{2\pi}{\alpha_n}}\,\mathrm{r}_n(x)\right)-\Theta\left(\varphi\left(-\frac{\log|x|}{\alpha_n}\right)\right)\right].$$

Since $\frac{|\psi(s)|+|\varphi(s)|}{\sqrt{s}}$ goes to zero as s tends to either 0 or ∞, there exists $0<s_0<S_0$ such that

$$|\psi(s)|+|\varphi(s)|\leq\varepsilon\sqrt{s},\quad s\in[0,s_0]\cup[S_0,\infty[.\tag{46}$$

We first take the case

$$\frac{\beta_n}{\alpha_n}\longrightarrow\infty\,.\tag{47}$$

For $\lambda>0$, we have

$$\begin{aligned}\int_{\mathbb{R}^2}\left(\mathrm{e}^{|\frac{g_n(x)}{\lambda}|^2}-1\right)dx&=\int_{\mathcal{C}_n}\left(\mathrm{e}^{|\frac{g_n(x)}{\lambda}|^2}-1\right)dx+\int_{\mathbb{R}^2\backslash\mathcal{C}_n}\left(\mathrm{e}^{|\frac{g_n(x)}{\lambda}|^2}-1\right)dx\\&=\mathbf{I}_n+\mathbf{J}_n,\end{aligned}$$

where $\mathcal{C}_n=\left\{|x|\leq\mathrm{e}^{-s_0\beta_n}\right\}$.

Notice, that for $x\in\mathbb{R}^2\backslash\mathcal{C}_n$, we have $\left|\sqrt{\frac{\beta_n}{2\pi}}\,\psi\left(-\frac{\log|x|}{\beta_n}\right)\right|\leq\frac{\varepsilon}{\sqrt{2\pi}}\sqrt{-\log|x|}$ and hence using that $|\Theta'|\leq 2$, we get for $x\in\mathbb{R}^2\backslash\mathcal{C}_n$:

$$|g_n(x)|\leq 2\frac{\varepsilon}{\sqrt{2\pi}}\sqrt{-\log|x|}+2|r_n(x)|.$$

Taking advantage of the fact that the elementary concentration $\sqrt{\frac{\beta_n}{2\pi}}\,\psi\left(-\frac{\log|x|}{\beta_n}\right)$ is supported in the unit ball B_1, we deduce that

$$\begin{aligned}\mathbf{J}_n&\leq\int_{B_1\backslash\mathcal{C}_n}\left(\mathrm{e}^{8\frac{\varepsilon^2}{2\pi\lambda^2}|\log|x||}-1\right)dx+\int_{\mathbb{R}^2\backslash\mathcal{C}_n}\left(\mathrm{e}^{8\frac{|r_n(x)|^2}{\lambda^2}}-1\right)dx\\&\leq\kappa,\end{aligned}$$

[2]For simplicity, we write Θ instead of Θ_a^M.

for n sufficiently large and ε small enough satisfying $8\varepsilon^2 < 2\pi\lambda^2$.

To control $\mathbf{I}_n$, we use that Θ is bounded:

$$\begin{aligned}\mathbf{I}_n &\le \int_{\mathcal{C}_n}\left(\mathrm{e}^{\frac{2M^2\alpha_n}{\pi\lambda^2}}-1\right)dx\\ &\le \pi\left(\mathrm{e}^{\frac{2M^2\alpha_n}{\pi\lambda^2}}-1\right)\mathrm{e}^{-2s_0\beta_n}\to 0.\end{aligned}$$

Now, we assume that

$$\frac{\beta_n}{\alpha_n}\longrightarrow 0\,. \tag{48}$$

For $\lambda>0$, we have

$$\begin{aligned}\int_{\mathbb{R}^2}\left(\mathrm{e}^{|\frac{g_n(x)}{\lambda}|^2}-1\right)dx &= \int_{\mathcal{D}_n}\left(\mathrm{e}^{|\frac{g_n(x)}{\lambda}|^2}-1\right)dx+\int_{\mathbb{R}^2\setminus\mathcal{D}_n}\left(\mathrm{e}^{|\frac{g_n(x)}{\lambda}|^2}-1\right)dx\\ &= \mathbf{I}_n+\mathbf{J}_n,\end{aligned}$$

where $\mathcal{D}_n=\Big\{|x|\le \mathrm{e}^{-S_0\beta_n}\Big\}$.

Notice, that for $x\in\mathcal{D}_n$, we have $\left|\sqrt{\frac{\beta_n}{2\pi}}\,\psi\left(-\frac{\log|x|}{\beta_n}\right)\right|\le\frac{\varepsilon}{\sqrt{2\pi}}\sqrt{-\log|x|}$ and hence using that $|\Theta'|\le 2$, we get for $x\in\mathcal{D}_n$:

$$|g_n(x)|\le 2\frac{\varepsilon}{\sqrt{2\pi}}\sqrt{-\log|x|}+2|r_n(x)|.$$

It follows that

$$\begin{aligned}\mathbf{I}_n &\le \int_{\mathcal{D}_n}\left(\mathrm{e}^{8\frac{\varepsilon^2}{2\pi\lambda^2}|\log|x||}-1\right)dx+\int_{\mathcal{D}_n}\left(\mathrm{e}^{8\frac{|r_n(x)|^2}{\lambda^2}}-1\right)dx\\ &\le \kappa,\end{aligned}$$

for n sufficiently big and ε small enough so that $8\varepsilon^2<2\pi\lambda^2$.

To control the integral $\mathbf{J}_n$, we use that for $x\in\mathbb{R}^2\setminus\mathcal{D}_n$ and n large enough, we have $\frac{-\log|x|}{\alpha_n}<S_0\frac{\beta_n}{\alpha_n}\to 0$. Thus

$$\left|\varphi\left(\frac{-\log|x|}{\alpha_n}\right)\right|\le\varepsilon\sqrt{\frac{-\log|x|}{\alpha_n}}\le\varepsilon\sqrt{\frac{S_0\beta_n}{\alpha_n}}.$$

Let $M_1=\sup_{s<S_0}|\psi(s)|$. Using the fact that Θ is non-decreasing, $|\Theta(s)|\le|s|$ and

$$|\Theta(s_1+s_2+s_3)|\le|\Theta(3s_1)|+|\Theta(3s_2)|+|\Theta(3s_3)|,$$

we infer

$$|g_n(x)|\le\sqrt{\frac{\alpha_n}{2\pi}}\left[2\Theta\Big(3\sqrt{\varepsilon}\sqrt{\frac{S_0\beta_n}{\alpha_n}}\Big)+\Theta\Big(3M_1\sqrt{\frac{\beta_n}{\alpha_n}}\Big)\right]+6|r_n(x)|=6|r_n(x)|,$$

for n large enough. The conclusion follows at least for $l = 2$.

To extend this proof to the case $l \geq 3$, we can without loss of generality assume that $\frac{\alpha_n^{(j)}}{\alpha_n^{(j+1)}} \to 0$ when n goes to infinity, $1 \leq j \leq l-1$. We also assume that $1 < k < l$ otherwise we need only one splitting as in the case $l = 2$. Now, we have just to combine the two previous cases by introducing

$$\mathcal{C}_n = \Big\{|x| \leq \mathrm{e}^{-s_0 \alpha_n^{(k+1)}}\Big\}, \quad \text{and} \quad \mathcal{D}_n = \Big\{|x| \leq \mathrm{e}^{-S_0 \alpha_n^{(k-1)}}\Big\},$$

where $0 < s_0 < S_0$ are such that

$$|\varphi^{(j)}(s)| \leq \varepsilon \sqrt{s}, \quad s \in [0, s_0] \cup [S_0, \infty[, \tag{49}$$

and split the integral into three parts: For $\lambda > 0$, we denote

$$\begin{aligned}
\int_{\mathbb{R}^2} \left(\mathrm{e}^{|\frac{g_n(x)}{\lambda}|^2} - 1\right) dx &= \int_{\mathcal{C}_n} \left(\mathrm{e}^{|\frac{g_n(x)}{\lambda}|^2} - 1\right) dx + \int_{\mathcal{D}_n \setminus \mathcal{C}_n} \left(\mathrm{e}^{|\frac{g_n(x)}{\lambda}|^2} - 1\right) dx \\
&+ \int_{\mathbb{R}^2 \setminus \mathcal{D}_n} \left(\mathrm{e}^{|\frac{g_n(x)}{\lambda}|^2} - 1\right) dx \\
&= \mathbf{I}_n + \mathbf{J}_n + \mathbf{K}_n.
\end{aligned}$$

The rest of the proof combines the previous two cases. We have, therefore, completed the proof of Lemma 2.4. □

Invoking Lebesgue theorem and Theorem 6.19 page 154 in [32], we find that $\Theta_a^M(\varphi^{(1)})$ goes to $\varphi^{(1)}$ in $\mathcal{P}$ when a goes to zero and M goes to infinity. Indeed

$$\begin{aligned}
\int_0^\infty \left|(\Theta_a^M)'\left(\varphi^{(1)}(s)\right) - 1\right|^2 \left|\varphi^{(1)\prime}(s)\right|^2 ds &= \int_{\{\varphi^{(1)}(s)=0\}} \left|\varphi^{(1)\prime}(s)\right|^2 ds \\
+ \int_{\{|\varphi^{(1)}(s)|>0\}} \left|(\Theta_a^M)'\left(\varphi^{(1)}(s)\right) - 1\right|^2 \left|\varphi^{(1)\prime}(s)\right|^2 ds\,.
\end{aligned}$$

The first integral vanishes since $\varphi^{(1)\prime}(s) = 0$ for almost s in the set $\{\varphi^{(1)}(s) = 0\}$, while the second integral can be dealt using the simple convergence of the sequence $\left((\Theta_a^M)'\left(\varphi^{(1)}(s)\right) - 1\right)$ to zero when (a, M) goes to $(0, \infty)$ and the fact that

$$\left|(\Theta_a^M)'\left(\varphi^{(1)}(s)\right) - 1\right|^2 \left|\varphi^{(1)\prime}(s)\right|^2 \lesssim \left|\varphi^{(1)\prime}(s)\right|^2.$$

Finally, as the function Θ_a^M vanishes at 0, we easily get

$$\int_0^\infty \left|\Theta_a^M\left(\varphi^{(1)}(s)\right) - \varphi^{(1)}(s)\right|^2 \mathrm{e}^{-2s}\, ds \to 0\,.$$

The proof of Proposition 2.3 is then achieved. □

2.4. **Extraction of the cores and profiles.**

2.4.1. *Extraction of the first core.* Due to the previous subsection and estimates (36)-(37), it is enough to make the extraction from $\widetilde{u}_{n,a}^{M}$. Ignoring the rest term, we are reduced to studying the case of a sequence u_n satisfying

$$u_n^*(x) = \sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\,\varphi^{(1)}\left(\frac{-\log|x|}{\alpha_n^{(1)}}\right) \tag{50}$$

and such that $|\varphi^{(1)}|$ is bounded by M. This assumption will only be made in this subsection. Our approach to extract cores and profiles relies on a diagonal subsequence extraction and the heart of the matter is reduced to the proof of the following lemma:

Lemma 2.5. *Under the above notation, there exist $\delta_0 > 0$ and $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$ there exists x_n such that*

$$\frac{|E_n \cap B(x_n, \mathrm{e}^{-b\,\alpha_n^{(1)}})|}{|E_n|} \geq \delta_0 A_0^2, \tag{51}$$

where $E_n := \left\{x \in \mathbb{R}^2;\ |u_n(x)| \geq \sqrt{2\alpha_n^{(1)}}\left(1 - \frac{\varepsilon_0}{10}\right) A_0\right\}$, $B(x_n, \mathrm{e}^{-b\,\alpha_n^{(1)}})$ designates the ball of center x_n and radius $\mathrm{e}^{-b\,\alpha_n^{(1)}}$ with $b = 1 - 2\varepsilon_0$ and $|\cdot|$ still denotes the Lebesgue measure.

Proof. Let us assume by contradiction that (51) does not hold. Consequently, up to a subsequence extraction, we have for any $\delta > 0$, $n \in \mathbb{N}$ and $x \in \mathbb{R}^2$

$$\frac{|E_n \cap B(x, \mathrm{e}^{-b\,\alpha_n^{(1)}})|}{|E_n|} \leq \delta A_0^2\,. \tag{52}$$

In particular, inequality (52) occurs for any ball centered at a point belonging to $\mathbf{T}_n := (\mathrm{e}^{-b\,\alpha_n^{(1)}}\mathbb{Z}) \times (\mathrm{e}^{-b\,\alpha_n^{(1)}}\mathbb{Z})$. It will be useful later on to notice that the balls $B(x, \mathrm{e}^{-b\,\alpha_n^{(1)}})$ constitute a covering of $\mathbb{R}^2$ when the point x varies in $\mathbf{T}_n$ and that each point of $\mathbb{R}^2$ belongs at most to four balls among

$$\mathcal{B}_n := \Big\{\, B(x, \mathrm{e}^{-b\,\alpha_n^{(1)}}),\ x \in \mathbf{T}_n \Big\}.$$

This implies in particular that

$$\|\nabla u_n\|^2_{L^2(\mathbb{R}^2)} \geq \frac{1}{4}\sum_{\mathbf{B}\in\mathcal{B}_n} \|\nabla u_n\|^2_{L^2(\mathbf{B})}\,. \tag{53}$$

Now, our goal is to get a contradiction by proving that for δ small enough the sum $\frac{1}{4}\sum_{\mathbf{B}\in\mathcal{B}_n} \|\nabla u_n\|^2_{L^2(\mathbf{B})}$ exceeds the energy of u_n.

For this purpose, let us first estimate the energy of u_n on each ball $\mathbf{B} \in \mathcal{B}_n$, making use of capacity arguments. To do so, we shall take advantage of the fact

that the values of $|u_n|$ on $\mathbf{B}$ varies at least from $\sqrt{2\alpha_n^{(1)}}\,\Big(1-\frac{\varepsilon_0}{10}\Big)\,A_0$ on $E_n\cap\mathbf{B}$ to $\sqrt{2\alpha_n^{(1)}}\,\Big(1-\frac{\varepsilon_0}{2}\Big)\,A_0$ on a set of Lebesgue measure greater than $\frac{|\mathbf{B}|}{2}$, for all $n\geq N_{\varepsilon_0}$, where N_{ε_0} is an integer big enough which only depends on ε_0.

Indeed, by definition of E_n, we have

$$|u_{n_{|E_n\cap\mathbf{B}}}|\geq\sqrt{2\alpha_n^{(1)}}\,\Big(1-\frac{\varepsilon_0}{10}\Big)\,A_0.$$

Besides, thanks to (41) and (42), we get that for any $s\leq 1-\varepsilon_0$

$$\varphi^{(1)}(s)\leq(1-\frac{\varepsilon_0}{2})\sqrt{4\pi}\,A_0. \tag{54}$$

Thus, if we designate by H_n^* the set

$$H_n^*:=\Big\{\;x\in\mathbb{R}^2;\quad u_n^*(x)\geq\sqrt{2\alpha_n^{(1)}}\,\Big(1-\frac{\varepsilon_0}{2}\Big)\,A_0\Big\},$$

we obtain in view of (50) and (54) that $H_n^*\subset B(0,\mathrm{e}^{-(1-\varepsilon_0)\alpha_n^{(1)}})$ which implies that $|H_n^*|\leq\pi\mathrm{e}^{-2(1-\varepsilon_0)\alpha_n^{(1)}}$. We deduce, by virtue of Proposition 3.10 that the set

$$H_n:=\Big\{\;x\in\mathbb{R}^2;\quad|u_n(x)|\;\geq\sqrt{2\alpha_n^{(1)}}\,\Big(1-\frac{\varepsilon_0}{2}\Big)\,A_0\Big\}$$

is of Lebesgue measure $|H_n|=|H_n^*|\leq\pi\mathrm{e}^{-2(1-\varepsilon_0)\alpha_n^{(1)}}$.

Since $|\mathbf{B}|=\pi\mathrm{e}^{-2(1-2\varepsilon_0)\alpha_n^{(1)}}$ and $\alpha_n^{(1)}\to\infty$ as n goes to infinity, there exists N_{ε_0} (which only depends on ε_0) such that the ball $\mathbf{B}$ contains a set $\widetilde{\mathbf{B}}_n$ on which we have $|u_n|\leq\sqrt{2\alpha_n^{(1)}}\,\Big(1-\frac{\varepsilon_0}{2}\Big)\,A_0$ and so that $|\widetilde{\mathbf{B}}_n|\geq\frac{|\mathbf{B}|}{2}$ for all $n\geq N_{\varepsilon_0}$.

To achieve the proof of Lemma 2.5 and get a contradiction, we will estimate the energy of u_n on the set $\widetilde{\mathcal{B}}_n$ of balls $\mathbf{B}\in\mathcal{B}_n$ satisfying $|E_n\cap\mathbf{B}|\geq\mathrm{e}^{-10\,\alpha_n^{(1)}}$. Indeed by virtue of (31) and as it was point out in Remarks 1.17, we can reduce to the case where concentrations occur only in a fixed ball $B(0,R_0)$. But, since we cover the ball $B(0,R_0)$ by at most a number of order $\mathrm{e}^{2\,b\,\alpha_n^{(1)}}$ of balls that are part of the set $\mathcal{B}_n$, necessary mass concentrated in E_n is mainly due to balls of $\widetilde{\mathcal{B}}_n$. In effect, the contribution of balls $\mathbf{B}\in\mathcal{B}_n$ satisfying $|E_n\cap\mathbf{B}|\leq\mathrm{e}^{-10\,\alpha_n^{(1)}}$ is at most equal to $\mathrm{e}^{-8\,\alpha_n^{(1)}}$ which is a negligible part of $|E_n|$ in view of Remark 2.2 and Proposition 3.10.

Taking advantage of the fact that the values of $|u_n|$ on $\mathbf{B}\in\widetilde{\mathcal{B}}_n$ varies at least from the value $\sqrt{2\alpha_n^{(1)}}\,\Big(1-\frac{\varepsilon_0}{10}\Big)\,A_0$ on the set $E_n\cap\mathbf{B}$ of Lebesgue measure greater than $\mathrm{e}^{-10\,\alpha_n^{(1)}}$ to the value $\sqrt{2\alpha_n^{(1)}}\,\Big(1-\frac{\varepsilon_0}{2}\Big)\,A_0$ on $\widetilde{\mathbf{B}}_n$ which is of Lebesgue measure

greater than $\frac{|\mathbf{B}|}{2}$ for $n \geq N_{\varepsilon_0}$, it follows from Proposition 3.18 that

$$\begin{aligned} \|\nabla u_n\|^2_{L^2(\mathbf{B})} &\geq C\Big((\frac{\varepsilon_0}{2} - \frac{\varepsilon_0}{10}) \sqrt{2\alpha_n^{(1)}}\; A_0 \Big)^2 \frac{1}{\log\Big(\frac{\mathrm{e}^{-(1-2\varepsilon_0)\alpha_n^{(1)}}}{\sqrt{|E_n \cap \mathbf{B}|}} \Big)} \\ &\geq C\varepsilon_0^2\, A_0^2\,, \qquad (55) \end{aligned}$$

where C is an absolute constant.

Hence, thanks to (53), we obtain

$$4\,\|\nabla u_n\|^2_{L^2(\mathbb{R}^2)} \geq \#(\widetilde{\mathcal{B}}_n)\, C\varepsilon_0^2\, A_0^2, \qquad (56)$$

where $\#(\widetilde{\mathcal{B}}_n)$ denotes the cardinal of $\widetilde{\mathcal{B}}_n$. But by (52), the covering of $\mathbb{R}^2$ by $\mathcal{B}_n$ and the fact that mass concentrated in E_n is mainly due to balls of $\widetilde{\mathcal{B}}_n$, we have necessary

$$\#(\widetilde{\mathcal{B}}_n) \geq \frac{1}{2\delta\, A_0^2}$$

which yields a contradiction for δ small enough in view of (56).

□

2.4.2. *Extraction of the first profile.* Let us set

$$\psi_n(y,\theta) = \sqrt{\frac{2\pi}{\alpha_n^{(1)}}}\; v_n(\alpha_n^{(1)} y, \theta), \qquad (57)$$

where $v_n(s,\theta) = \Big(\tau_{-x_n^{(1)}} u_n\Big)(\mathrm{e}^{-s}\cos\theta, \mathrm{e}^{-s}\sin\theta)$ and $x_n^{(1)}$ satisfies

$$\frac{|E_n \cap B(x_n^{(1)}, \mathrm{e}^{-(1-2\varepsilon_0)\,\alpha_n^{(1)}})|}{|E_n|} \geq \delta_0 A_0^2, \qquad (58)$$

with E_n defined by Lemma 2.5. Using the invariance of Lebesgue measure under translations, we get

(59)

$$\|\nabla u_n\|^2_{L^2} = \frac{1}{2\pi}\int_{\mathbb{R}}\int_0^{2\pi} |\partial_y \psi_n(y,\theta)|^2 dy d\theta + \frac{(\alpha_n^{(1)})^2}{2\pi}\int_{\mathbb{R}}\int_0^{2\pi} |\partial_\theta \psi_n(y,\theta)|^2 dy d\theta.$$

Since $\alpha_n^{(1)}$ tends to infinity and (u_n) is bounded in $H^1(\mathbb{R}^2)$, (59) implies that

$$\begin{aligned} \partial_\theta \psi_n &\to 0 \quad \text{and} \qquad (60) \\ \partial_y \psi_n &\rightharpoonup g, \qquad (61) \end{aligned}$$

up to a subsequence extraction, in $L^2(y,\theta)$ as n tends to infinity. Moreover

$$\psi_n \to 0 \text{ in } L^2(]-\infty, 0] \times [0, 2\pi]) \text{ as } n \to \infty. \qquad (62)$$

Indeed, we have

$$\begin{aligned}\|u_n\|^2_{L^2} &= \int_{\mathbb{R}}\int_0^{2\pi} |v_n(s,\theta)|^2 \mathrm{e}^{-2s}\, ds\, d\theta \\ &= \frac{(\alpha_n^{(1)})^2}{2\pi}\int_{\mathbb{R}}\int_0^{2\pi} |\psi_n(y,\theta)|^2 \mathrm{e}^{-2\alpha_n^{(1)} y}\, dy\, d\theta \\ &\geq \frac{(\alpha_n^{(1)})^2}{2\pi}\int_{-\infty}^0\int_0^{2\pi} |\psi_n(y,\theta)|^2\, dy\, d\theta,\end{aligned}$$

which ends the proof of (62).

The following lemma summarizes the principle properties of the function g given above by (61).

Lemma 2.6. *The function g given by* (61) *only depends on the variable y and is null on $]-\infty,0]$. Besides, we have*

$$\frac{1}{2\pi}\int_0^{2\pi}\Big(\psi_n(y_2,\theta)-\psi_n(y_1,\theta)\Big)\, d\theta \to \int_{y_1}^{y_2} g(y)\, dy, \tag{63}$$

for any $y_1, y_2 \in \mathbb{R}$, as n tends to infinity.

Proof of Lemma 2.6. Let us go to the proof of the fact that the function g only depends on the variable y. First, by (60), $\partial_\theta\psi_n \to 0$ in $L^2(y,\theta)$ and hence we have $\partial_y(\partial_\theta\psi_n) \to 0$ in $\mathcal{D}'$. Second, under (61), $\partial_y\psi_n \to g$ in $\mathcal{D}'$ which implies that $\partial_\theta(\partial_y\psi_n) \to \partial_\theta g$ in $\mathcal{D}'$. Therefore, we deduce that $\partial_\theta g = 0$. The fact that $g \equiv 0$ on $]-\infty,0]$ derives from (62).

Now, taking advantage of the fact that $\partial_y\psi_n \rightharpoonup g$ in L^2 as n tends to infinity, we get for any $y_1 \leq y_2$,

$$\langle \partial_y\psi_n, \mathbf{1}_{[y_1,y_2]}\rangle \to 2\pi\int_{y_1}^{y_2} g(y)\, dy.$$

But,

$$\begin{aligned}\langle \partial_y\psi_n, \mathbf{1}_{[y_1,y_2]}\rangle &= \int_0^{2\pi}\int_{y_1}^{y_2} \partial_y\psi_n(y,\theta)\, dy\, d\theta \\ &= \int_0^{2\pi} (\psi_n(y_2,\theta)-\psi_n(y_1,\theta))\, d\theta.\end{aligned}$$

This leads to (63). □

Before extracting the first profile, let us begin by establish the following lemma:

Lemma 2.7. *The function $F_n(y) = \frac{1}{2\pi}\int_0^{2\pi} \psi_n(y,\theta)\, d\theta$ is continuous on $\mathbb{R}$.*

Proof of Lemma 2.7. By Cauchy-Schwarz's inequality, we get for y_1 and y_2 in $\mathbb{R}$:

$$\left|F_n(y_1) - F_n(y_2)\right|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} \left|\psi_n(y_1,\theta) - \psi_n(y_2,\theta)\right|^2 d\theta.$$

But,

$$\begin{aligned}\Big|\psi_n(y_1,\theta) - \psi_n(y_2,\theta)\Big| &= \Big|\int_{y_1}^{y_2} \partial_y\psi_n(\tau,\theta)\,d\tau\Big| \\ &\leq \sqrt{|y_1-y_2|}\Big(\int_{y_1}^{y_2} \left|\partial_y\psi_n(\tau,\theta)\right|^2 d\tau\Big)^{\frac{1}{2}}\end{aligned}$$

Therefore

$$\left|F_n(y_1) - F_n(y_2)\right|^2 \leq \frac{1}{2\pi}|y_1-y_2| \int_0^{2\pi}\int_{y_1}^{y_2} \left|\partial_y\psi_n(\tau,\theta)\right|^2 d\tau\,d\theta \leq C|y_1-y_2|,$$

which implies that the sequence (F_n) is uniformly in n in the Hölder space $C^{\frac{1}{2}}(\mathbb{R})$. □

Let us now introduce the function

$$\psi^{(1)}(y) = \int_0^y g(\tau)\,d\tau. \tag{64}$$

Our goal in what follows is to prove the following proposition:

Proposition 2.8. *The function $\psi^{(1)}$ defined by* (64) *belongs to the set of profiles $\mathcal{P}$. Besides for any $y \in \mathbb{R}$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi} \psi_n(y,\theta)\,d\theta \to \psi^{(1)}(y), \tag{65}$$

as n tends to infinity and there exists an absolute constant C so that

$$\|\psi^{(1)'}\|_{L^2} \geq C A_0\,. \tag{66}$$

Remark 2.9. *Let us point out that by virtue of the convergence properties* (60) *and* (61)*, the sequence $(\frac{1}{2\pi}\psi_n(y,\theta))$ converges weakly to $\psi^{(1)}(y)$ in $L^2(\mathbb{R}\times[0,2\pi])$ as n tends to infinity. In particular we have for any $f \in \mathcal{D}(\mathbb{R})$*

$$\frac{1}{2\pi}\int_{\mathbb{R}}\int_0^{2\pi} \psi_n(y,\theta) f(y)\,dy\,d\theta \overset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}} \psi^{(1)}(y) f(y)\,dy.$$

In other words

$$\frac{1}{2\pi}\int_{\mathbb{R}}\int_0^{2\pi} \sqrt{\frac{2\pi}{\alpha_n^{(1)}}}\,\Big(\tau_{-x_n^{(1)}}u_n\Big)(\mathrm{e}^{-\alpha_n^{(1)}y}\cos\theta, \mathrm{e}^{-\alpha_n^{(1)}y}\sin\theta) f(y)\,dy\,d\theta \overset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}} \psi^{(1)}(y) f(y)\,dy. \tag{67}$$

Proof of Proposition 2.8. Clearly $\psi^{(1)} \in \mathcal{C}(\mathbb{R})$ and $\psi^{(1)'} = g \in L^2(\mathbb{R})$. Moreover, since

$$\left|\psi^{(1)}(y)\right| = \Big|\int_0^y g(\tau)\,d\tau\Big| \leq \sqrt{y}\,\|g\|_{L^2(\mathbb{R})},$$

we get $\psi^{(1)} \in L^2(\mathbb{R}^+, \mathrm{e}^{-2y}\,dy)$. Now, using the fact that $g \equiv 0$ on $\mathbb{R}^-$, we get the assertion (65). Indeed, by virtue of (63), it suffices to prove that for any $y \leq 0$,

$$F_n(y) = \frac{1}{2\pi}\int_0^{2\pi} \psi_n(y,\theta)\,d\theta \to 0, \quad n \to \infty\,.$$

Applying Cauchy-Schwarz's inequality and integrating with respect to y, we obtain

$$\int_{-\infty}^{0} \Big|F_n(y)\Big|^2\,dy \leq \frac{1}{2\pi}\,\|\psi_n\|^2_{L^2(]-\infty,0]\times[0,2\pi])},$$

which implies by virtue of (62) that the sequence (F_n) converges strongly to 0 in $L^2(]-\infty,0])$. Therefore, up to a subsequence extraction

$$F_n(y) \to 0 \text{ almost every where in }]-\infty, 0]\,.$$

Taking advantage of the continuity of F_n, we deduce that for any $y \leq 0$, $F_n(y) \to 0$ which achieves the proof of claim (65).

To end the proof of Proposition 2.8, it remains to check (66) which is the key estimate to iterate the process of extraction of elementary concentrations. By virtue of (65) and Cauchy-Schwarz's inequality

$$\Big|\psi^{(1)}(y)\Big| = \Big|\int_0^y \psi^{(1)\prime}(\tau)\,d\tau\Big| \leq \sqrt{y}\,\|\psi^{(1)\prime}\|_{L^2}.$$

So to establish the key estimate (66), it suffices in light of (65) to prove the existence of y_0 close to 1 such that for n big enough

$$\Big|\frac{1}{2\pi}\int_0^{2\pi} \psi_n(y_0,\theta)\,d\theta\Big| \geq C A_0\,, \tag{68}$$

where C is an absolute constant. For this purpose, we will again use capacity arguments. First, we define

$$E_n^{\pm} := \Big\{x \in \mathbb{R}^2;\ \pm u_n(x)\ \geq \sqrt{2\alpha_n^{(1)}}\,\Big(1-\frac{\varepsilon_0}{10}\Big)\,A_0\Big\}.$$

Hence, $E_n = E_n^+ \cup E_n^-$. Modulo replacing u_n by $-u_n$, we can assume that (51) yields

$$|E_n^+ \cap B(x_n^{(1)}, \mathrm{e}^{-(1-2\,\varepsilon_0)\,\alpha_n^{(1)}})| \geq \frac{\delta_0}{2}A_0^2|E_n| \geq \frac{\delta_0}{2}A_0^2\mathrm{e}^{-2\,\alpha_n^{(1)}}. \tag{69}$$

We also write $u_n = u_n^+ - u_n^-$ and $\psi_n = \psi_n^+ - \psi_n^-$ where $u_n^+, u_n^-, \psi_n^+, \psi_n^- \geq 0$. Introducing the set $\widetilde{E_n} \supset E_n^+$ defined by

$$\widetilde{E_n} := \Big\{x \in \mathbb{R}^2;\ u_n^+(x) \geq \frac{\sqrt{2\alpha_n^{(1)}}A_0}{2}\Big\},$$

we infer that we can choose ε_0 so that for n big enough

$$\Big|\widetilde{E_n} \cap B(x_n^{(1)}, \mathrm{e}^{-(1-2\,\varepsilon_0)\,\alpha_n^{(1)}})\Big| \geq \frac{1}{2}\Big|B(x_n^{(1)}, \mathrm{e}^{-(1-2\,\varepsilon_0)\,\alpha_n^{(1)}})\Big|\,. \tag{70}$$

Otherwise taking into account the fact that the values of the function u_n^+ on the ball $B(x_n^{(1)}, \mathrm{e}^{-(1-2\,\varepsilon_0)\,\alpha_n^{(1)}})$, that we shall designate in what follows by $\mathbf{B}$ to avoid heaviness, varies at least from values larger than $\sqrt{2\alpha_n^{(1)}}\,\left(1-\frac{\varepsilon_0}{10}\right)A_0$ on $E_n^+\cap\mathbf{B}$ to values smaller than $\frac{1}{2}\sqrt{2\alpha_n^{(1)}}\,A_0$ on a subset of $\mathbf{B}$ of Lebesgue measure greater than $\frac{|\mathbf{B}|}{2}$, it comes from Proposition 3.18 that

$$\|\nabla u_n\|^2_{L^2(\mathbf{B})} \geq \|\nabla u_n^+\|^2_{L^2(\mathbf{B})} \geq 2\pi\Big(\Big(\frac{1}{2}-\frac{\varepsilon_0}{10}\Big)\sqrt{2\alpha_n^{(1)}}\;A_0\Big)^2\frac{1}{\log\frac{\mathrm{e}^{-\alpha_n^{(1)}(1-2\varepsilon_0)}}{\sqrt{|E_n^+\cap\mathbf{B}|}}}\,.$$

Therefore, we get by virtue of Lemma 2.5 and (69) that

$$\|\nabla u_n\|^2_{L^2(\mathbf{B})} \geq \frac{CA_0^2}{\varepsilon_0}\,,$$

for n big enough which gives a contradiction. Hence (70) holds.

Since the measure of the ball $B(x_n^{(1)}, \mathrm{e}^{-(1+2\,\varepsilon_0)\,\alpha_n^{(1)}})$ is much smaller than the measure of $\mathbf{B}$, we deduce the existence of y_0 such that $1-2\,\varepsilon_0\leq y_0\leq 1+2\,\varepsilon_0$ and $\psi_n^+(y_0,\theta)\geq\sqrt{\pi}\,A_0$ for θ varying over an interval of length at least π. Hence, the limit ψ^+ of ψ_n^+ satisfies

$$\psi^+(y_0)\geq\frac{\sqrt{\pi}}{2}\,A_0\,. \tag{71}$$

Now, we argue in a similar way to control $\psi^-(y_0)$ from above. Let $\tilde{E}_n^-$ be the set where $u_n^->0$. The values of u_n on the ball $\mathbf{B}$ vary from values larger than $\frac{1}{2}\sqrt{2\alpha_n^{(1)}}A_0$ on $E_n^+\cap\mathbf{B}$ to negative values on $\tilde{E}_n^-\cap\mathbf{B}$. Using Proposition 3.18, we deduce that

$$\|\nabla u_n\|^2_{L^2(\mathbf{B})} \geq 2\pi\Big(\frac{1}{2}\sqrt{2\alpha_n^{(1)}}\;A_0\Big)^2\frac{1}{\log\frac{\sqrt{|\mathbf{B}|}}{\sqrt{|\tilde{E}_n^-\cap\mathbf{B}|}}}\,,$$

which implies that $|\tilde{E}_n^-\cap\mathbf{B}|\lesssim|\mathbf{B}|e^{-\pi\alpha_n^{(1)}A_0^2}$.

If $\tilde{E}_n^-\cap\mathbf{B}\subset B(x_n^{(1)}, \mathrm{e}^{-(1+2\,\varepsilon_0)\,\alpha_n^{(1)}})$, we are done. If not, there exists y_1 such that $1-2\,\varepsilon_0\leq y_1\leq 1+2\,\varepsilon_0$ and $\psi_n^-(y_1,\theta)>0$ for θ varying over an interval of length of measure at most $\frac{A_0}{10M}$ and n sufficiently large. As by construction $\|\psi_n\|_{L^\infty}\leq M$, the limit ψ^- of ψ_n^- satisfies $\psi^-(y_1)\leq\frac{1}{2\pi}\frac{A_0}{10}$. Since, ψ^- belongs to the Hölder space $C^{1/2}(\mathbb{R})$ and $|y_0-y_1|\leq 2\varepsilon_0$, we deduce that

$$\psi^-(y_0)\leq\frac{1}{2\pi}\frac{A_0}{10}+C\varepsilon_0^{1/2}\leq\frac{A_0}{10\pi}\,, \tag{72}$$

if ε_0 was chosen small enough. Finally combining (71) and (72), we infer that

$$\psi(y_0)=\psi^+(y_0)-\psi^-(y_0)\geq\frac{A_0}{2}$$

which achieves the proof of the last point of Proposition 2.8. □

2.5. **Iteration.** Our concern now is to iterate the previous process and to prove that the algorithmic construction converges. We do not assume anymore that u^n has only one sacle. Setting

$$\mathrm{r}_n^{(1)}(x)=\sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\,\left(\psi_n\Big(\frac{-\log|x-x_n^{(1)}|}{\alpha_n^{(1)}},\theta\Big)-\psi^{(1)}\Big(\frac{-\log|x-x_n^{(1)}|}{\alpha_n^{(1)}}\Big)\right),$$

where $(x_n^{(1)})$ is the sequence of points defined by (58) and

$$\psi_n(y,\theta)=\sqrt{\frac{2\pi}{\alpha_n^{(1)}}}\,\Big(\tau_{-x_n^{(1)}}u_n\Big)\Big(\mathrm{e}^{-\alpha_n^{(1)}y}\cos\theta,\mathrm{e}^{-\alpha_n^{(1)}y}\sin\theta\Big).$$

Let us first prove that the sequence $(\mathrm{r}_n^{(1)})$ satisfies the hypothesis of Theorem 1.16.

By definition $\mathrm{r}_n^{(1)}(x)=u_n(x)-g_n^{(1)}(x)$, where $g_n^{(1)}(x)=\sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\,\psi^{(1)}(\frac{-\log|x-x_n^{(1)}|}{\alpha_n^{(1)}})$. Noticing that $g_n^{(1)}\rightharpoonup 0\,\text{in}\,H^1$ when n tends to infinity, it is clear in view of (29) that $\mathrm{r}_n^{(1)}$ converges weakly to 0 as n tends to infinity.

On the other hand, thanks to the invariance of Lebesgue measure under translations, we have

$$\|\nabla u_n\|_{L^2}^2=\frac{1}{2\pi}\int_{\mathbb{R}}\int_0^{2\pi}|\partial_y\psi_n(y,\theta)|^2dyd\theta+\frac{(\alpha_n^{(1)})^2}{2\pi}\int_{\mathbb{R}}\int_0^{2\pi}|\partial_\theta\psi_n(y,\theta)|^2dyd\theta$$

and

$$\|\nabla g_n^{(1)}\|_{L^2}^2=\|\psi^{(1)\prime}\|_{L^2}^2.$$

Therefore

$$\|\nabla\mathrm{r}_n^{(1)}\|_{L^2}^2=\|\nabla u_n\|_{L^2}^2+\|\psi^{(1)\prime}\|_{L^2}^2-2\left(\frac{1}{2\pi}\int_{\mathbb{R}}\int_0^{2\pi}\partial_y\psi_n(y,\theta)\psi^{(1)\prime}(y)\,dyd\theta\right).$$

Taking advantage of the fact that by (61)

$$\partial_y\psi_n\rightharpoonup\psi^{(1)\prime},\quad\text{as}\quad n\to\infty\quad\text{in}\quad L^2(y,\theta),$$

we get

$$\lim_{n\to\infty}\|\nabla\,\mathrm{r}_n^{(1)}\|_{L^2}^2=\lim_{n\to\infty}\|\nabla\,u_n\|_{L^2}^2-\|\psi^{(1)\prime}\|_{L^2}^2\tag{73}$$

which implies that $(\mathrm{r}_n^{(1)})$ is bounded in $H^1(\mathbb{R}^2)$.

Now since $\psi^{(1)}_{|]-\infty,0]}=0$, we obtain for $R\geq 1$

$$\|\mathrm{r}_n^{(1)}\|_{\mathcal{L}(|x-x_n^{(1)}|\geq R)}=\|u_n\|_{\mathcal{L}(|x-x_n^{(1)}|\geq R)}.$$

Taking into account of the fact that the sequence $(x_n^{(1)})$ is bounded as it was observed in Remarks 1.17, we deduce that $(\mathrm{r}_n^{(1)})$ satisfies the hypothesis of compactness at infinity (31) and so $(\mathrm{r}_n^{(1)})$ verifies the hypothesis of Theorem 1.16.

Let us then define $A_1 = \limsup_{n\to\infty} \|\mathrm{r}_n^{(1)}\|_{\mathcal{L}}$. If $A_1 = 0$, we stop the process. If not, we apply the above arguments to $\mathrm{r}_n^{(1)}$ and then along the same lines as in Subsections 2.2, 2.3 and 2.4, there exist a scale $(\alpha_n^{(2)})$, a core $(x_n^{(2)})$ and a profile $\psi^{(2)}$ in $\mathcal{P}$ such that

$$\mathrm{r}_n^{(1)}(x) = \sqrt{\frac{\alpha_n^{(2)}}{2\pi}}\,\psi^{(2)}\left(\frac{-\log|x-x_n^{(2)}|}{\alpha_n^{(2)}}\right) + \mathrm{r}_n^{(2)}(x),$$

with $\|\psi^{(2)\prime}\|_{L^2} \geq C\,A_1$, C being the absolute constant appearing in (66). This leads to the following crucial estimate

$$\limsup_{n\to\infty} \|\mathrm{r}_n^{(2)}\|_{H^1}^2 \lesssim 1 - A_0^2 - A_1^2.$$

Moreover, we claim that $(\alpha_n^{(1)}, x_n^{(1)}, \psi^{(1)}) \perp (\alpha_n^{(2)}, x_n^{(2)}, \psi^{(2)})$ in the sense of the terminology introduced in Definition 1.15. In fact, if $\alpha_n^{(1)} \perp \alpha_n^{(2)}$, we are done. If not, by virtue of (14), we can suppose that $\alpha_n^{(1)} = \alpha_n^{(2)} = \alpha_n$ and our purpose is then to prove that

$$-\frac{\log|x_n^{(1)}-x_n^{(2)}|}{\alpha_n} \longrightarrow a \text{ as } n\to\infty,\ \text{with}\,\psi^{(1)} \text{ or } \psi^{(2)} \text{ null for } s<a\,. \tag{74}$$

First of all assuming that $-\frac{\log|x_n^{(1)}-x_n^{(2)}|}{\alpha_n} \to a$, let us prove that the profile $\psi^{(2)}$ is null for $s < a$. For this purpose, let us begin by recalling that in view of (67) we have for any $f \in \mathcal{D}(\mathbb{R})$,

$$\sqrt{\frac{1}{2\pi\alpha_n}}\int_{\mathbb{R}}\int_0^{2\pi} \mathrm{r}_n^{(1)}\Big(x_n^{(2)} + (\mathrm{e}^{-\alpha_n t}\cos\lambda, \mathrm{e}^{-\alpha_n t}\sin\lambda)\Big) f(t)dt\,d\lambda \to \int_{\mathbb{R}} \psi^{(2)}(t)f(t)\,dt,$$

as n tends to infinity.

Consequently for fixed $\varepsilon > 0$ we are reduced to demonstrate that for any function $f \in \mathcal{D}(]-\infty, a-\varepsilon[)$, we have

$$\lim_{n\to\infty}\sqrt{\frac{1}{2\pi\alpha_n}}\int_{-\infty}^{a-\varepsilon}\int_0^{2\pi} \mathrm{r}_n^{(1)}\Big(x_n^{(2)} + (\mathrm{e}^{-\alpha_n t}\cos\lambda, \mathrm{e}^{-\alpha_n t}\sin\lambda)\Big) f(t)\,dt\,d\lambda = 0. \tag{75}$$

To do so, let us perform the change of variables

$$(\mathrm{e}^{-\alpha_n s}\cos\theta, \mathrm{e}^{-\alpha_n s}\sin\theta) = x_n^{(2)} - x_n^{(1)} + (\mathrm{e}^{-\alpha_n t_n(s,\theta)}\cos\lambda_n(s,\theta), \mathrm{e}^{-\alpha_n t_n(s,\theta)}\sin\lambda_n(s,\theta)). \tag{76}$$

Denoting by $J_n(s,\theta)$ the Jacobian of the change of variables (76), we claim that

$$t_n(s,\theta) \overset{n\to\infty}{\longrightarrow} s, \tag{77}$$

and

$$J_n(s,\theta) \overset{n\to\infty}{\longrightarrow} 1, \tag{78}$$

uniformly with respect to $(s,\theta) \in]-\infty, a-\frac{\varepsilon}{2}] \times [0,2\pi]$.

Indeed, firstly we can observe that

$$\mathrm{e}^{-\alpha_n t_n(s,\theta)} = \mathrm{e}^{-\alpha_n s}|1+\Theta_n(s,\theta)| = \mathrm{e}^{-\alpha_n s}(1+o(1)), \tag{79}$$

where $\Theta_n(s,\theta) = \mathrm{e}^{\alpha_n s}\mathrm{e}^{-i\theta} z_n$, with z_n the writing in $\mathbb{C}$ of the point $x_n^{(1)} - x_n^{(2)}$. But by hypothesis, we know that $-\frac{\log|x_n^{(1)}-x_n^{(2)}|}{\alpha_n} \overset{n\to\infty}{\longrightarrow} a$. Then, there exists an integer N such that for all $n > N$,

$$\mathrm{e}^{-\alpha_n(a+\frac{\varepsilon}{4})} \le |x_n^{(1)} - x_n^{(2)}| \le \mathrm{e}^{-\alpha_n(a-\frac{\varepsilon}{4})} \tag{80}$$

which according to (76) and the fact that $t \in]-\infty, a-\varepsilon]$ gives rise (for $n > N$) to

$$\mathrm{e}^{-\alpha_n\, t_n(s,\theta)}(1-\mathrm{e}^{-\frac{\alpha_n\varepsilon}{4}}) \le \mathrm{e}^{-\alpha_n\, s} \le \mathrm{e}^{-\alpha_n\, t_n(s,\theta)}(1+\mathrm{e}^{-\frac{\alpha_n\varepsilon}{4}}).$$

This implies that $s \le a - \frac{\varepsilon}{2}$ for n big enough, and leads to (79). We deduce that

$$t_n(s,\theta) = s - \frac{\log|1+\Theta_n(s,\theta)|}{\alpha_n}, \tag{81}$$

with $|\Theta_n(s,\theta)| \le \mathrm{e}^{-\frac{\alpha_n\varepsilon}{4}}$ for $s \le a-\frac{\varepsilon}{2}$ and n sufficiently large, which achieves the proof of (77). Now, since $\frac{\partial\Theta_n}{\partial s} = \alpha_n\Theta_n$ and $\frac{\partial\Theta_n}{\partial\theta} = -i\Theta_n$, it follows that

$$\Big|\frac{\partial\Theta_n}{\partial s}\Big| \le \alpha_n \mathrm{e}^{-\frac{\alpha_n\varepsilon}{4}} \quad \text{and} \quad \Big|\frac{\partial\Theta_n}{\partial\theta}\Big| \le \mathrm{e}^{-\frac{\alpha_n\varepsilon}{4}},$$

which easily implies that

$$\frac{\partial t_n}{\partial s} = 1+o(1) \quad \text{and} \quad \frac{\partial t_n}{\partial\theta} = o(1), \tag{82}$$

where $|o(1)| \lesssim \mathrm{e}^{-\frac{\alpha_n\varepsilon}{4}}$ for $s \le a-\frac{\varepsilon}{2}$ and n big enough.

Otherwise, taking account of (76) and (81), we get

$$\mathrm{e}^{i\,\lambda_n(s,\theta)} = \frac{(1+\Theta_n(s,\theta))}{|1+\Theta_n(s,\theta)|}\,\mathrm{e}^{i\,\theta},$$

which allows by straightforward computations to prove that

$$\frac{\partial\lambda_n}{\partial s} = o(1) \quad \text{and} \quad \frac{\partial\lambda_n}{\partial\theta} = 1+o(1), \tag{83}$$

again with $|o(1)| \lesssim \mathrm{e}^{-\frac{\alpha_n\varepsilon}{4}}$ for $s \le a-\frac{\varepsilon}{2}$ and n big enough. Finally, the combination of (82) and (83) ensures claim (78).

Now, making the change of variables (76), the left hand side of (75) becomes for n big enough

$$I_n := \sqrt{\frac{1}{2\pi\alpha_n}} \int_{-\infty}^{a-\frac{\varepsilon}{2}} \int_0^{2\pi} \mathrm{r}_n^{(1)}\Big(x_n^{(1)} + (\mathrm{e}^{-\alpha_n\, s}\cos\theta, \mathrm{e}^{-\alpha_n\, s}\sin\theta)\Big) f(t_n(s,\theta)) J_n(s,\theta)\, ds\, d\theta.$$

But, by definition

$$\mathrm{r}_n^{(1)}(x) = u_n(x) - \sqrt{\frac{\alpha_n}{2\pi}}\,\psi^{(1)}\left(\frac{-\log|x-x_n^{(1)}|}{\alpha_n}\right).$$

Thus

$$I_n = \frac{1}{2\pi}\int_{-\infty}^{a-\frac{\varepsilon}{2}}\int_0^{2\pi}\Big(\sqrt{\frac{2\pi}{\alpha_n}}u_n\big(x_n^{(1)}+(\mathrm{e}^{-\alpha_n s}\cos\theta,\mathrm{e}^{-\alpha_n s}\sin\theta)\big)-\psi^{(1)}(s)\Big)f(t_n(s,\theta))J_n(s,\theta)dsd\theta.$$

Remembering that in view of (67) we have for any $f\in\mathcal{D}(\mathbb{R})$

$$\sqrt{\frac{1}{2\pi\alpha_n}}\int_{\mathbb{R}}\int_0^{2\pi}u_n\big(x_n^{(1)}+(\mathrm{e}^{-\alpha_n s}\cos\theta,\mathrm{e}^{-\alpha_n s}\sin\theta)\big)f(s)dsd\theta\overset{n\to\infty}{\longrightarrow}\int_0^\infty\psi^{(1)}(s)f(s)ds,$$

it comes under the fact that $t_n(s,\theta)\overset{n\to\infty}{\longrightarrow} s$ and $J_n(s,\theta)\overset{n\to\infty}{\longrightarrow} 1$ uniformly with respect to (s,θ) that

$$I_n\overset{n\to\infty}{\longrightarrow}\frac{1}{2\pi}\int_0^{a-\frac{\varepsilon}{2}}\int_0^{2\pi}\Big(\psi^{(1)}(s)-\psi^{(1)}(s)\Big)f(s)\,ds\,d\theta=0,$$

which concludes the proof of the fact that $\psi^{(2)}$ is null for $s<a$.

To achieve the proof of (74), it remains to show that the sequence

$$t_n := \frac{\log|x_n^{(1)}-x_n^{(2)}|}{\alpha_n}$$

is bounded and so, up to a subsequence extraction, it converges. Indeed, on one hand by virtue of Remark 1.17 the sequences of cores $(x_n^{(1)})$ and $(x_n^{(2)})$ are bounded which ensures the existence of a positive constant M such that $t_n\le\frac{\log M}{\alpha_n}$. On the other hand, there is $\gamma>0$ such that

$$|x_n^{(1)}-x_n^{(2)}|\ge\mathrm{e}^{-\gamma\alpha_n},\tag{84}$$

for n big enough. In effect, by construction $x_n^{(2)}$ is chosen so that there exist an absolute constant C and $t_0>0$ such that

$$\frac{1}{2\pi}\left|\int_0^{2\pi}\sqrt{\frac{2\pi}{\alpha_n}}\left(\tau_{-x_n^{(2)}}\mathrm{r}_n^{(1)}\right)(\mathrm{e}^{-\alpha_n t_0}\cos\lambda,\mathrm{e}^{-\alpha_n t_0}\sin\lambda)\,d\lambda\right|\ge CA_1\,,$$

for n sufficiently large, which can be written by designating $x_n^{(1)}-x_n^{(2)}$ by w_n

$$\frac{1}{2\pi}\left|\int_0^{2\pi}\Big(\psi_n\Big(\frac{-\log|\mathrm{e}^{-\alpha_n t_0}\mathrm{e}^{\mathrm{i}\lambda}-w_n|}{\alpha_n},\lambda\Big)-\psi^{(1)}\Big(\frac{-\log|\mathrm{e}^{-\alpha_n t_0}\mathrm{e}^{\mathrm{i}\lambda}-w_n|}{\alpha_n}\Big)\Big)\,d\lambda\right||\ge CA_1\,,$$

and implies that $|\psi^{(2)}(t_0)|\ge CA_1$. Since $\psi^{(2)}$ is continuous, there exists $\delta>0$ such that for $t\in[t_0-\delta,t_0+\delta]$, we have

$$|\psi^{(2)}(t)|\ge\frac{C}{2}A_1\,.$$

According to (67), we deduce that if f is a positive valued function belonging to $\mathcal{D}([t_0-\delta, t_0+\delta])$ and identically equal to one in $[t_0-\frac{\delta}{2}, t_0+\frac{\delta}{2}]$, then for n large enough

(85)
$$\Big|\sqrt{\frac{1}{2\pi\alpha_n}}\int_{t_0-\delta}^{t_0+\delta}\int_0^{2\pi} \mathrm{r}_n^{(1)}\Big(x_n^{(2)}+(\mathrm{e}^{-\alpha_n t}\cos\lambda, \mathrm{e}^{-\alpha_n t}\sin\lambda)\Big)f(t)dt\,d\lambda\Big| \geq \frac{C\,\delta}{2}A_1\,.$$

Now if claim (84) does not hold, then $\mathrm{e}^{-\alpha_n(t_0-2\delta)}w_n \to 0$. This yields along the same lines as above making use of the change of variables (76) to

$$\sqrt{\frac{1}{2\pi\alpha_n}}\int_{t_0-\delta}^{t_0+\delta}\int_0^{2\pi} \mathrm{r}_n^{(1)}\Big(x_n^{(2)}+(\mathrm{e}^{-\alpha_n t}\cos\lambda, \mathrm{e}^{-\alpha_n t}\sin\lambda)\Big)f(t)dt\,d\lambda \to 0\,,$$

which contradicts (85). Thus, there exists an integer N_1 such that for all $n \geq N_1$

$$-\gamma \leq t_n \leq \frac{\log M}{\alpha_n} \tag{86}$$

and then if ℓ denotes the limit of the sequence (t_n) (up to a subsequence extraction), we have necessary $\ell = -a$ with $0 \leq a \leq \gamma$. This ends the proof of the orthogonality property (74).

Finally, by iterating the process we get at step ℓ

$$u_n(x) = \sum_{j=1}^{\ell}\sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\,\psi^{(j)}\left(\frac{-\log|x-x_n^{(j)}|}{\alpha_n^{(j)}}\right)+\mathrm{r}_n^{(\ell)}(x),$$

with

$$\limsup_{n\to\infty}\|\mathrm{r}_n^{(\ell)}\|_{H^1}^2 \lesssim 1-A_0^2-A_1^2-\cdots-A_{\ell-1}^2\,.$$

Therefore $A_\ell \to 0$ as $\ell \to \infty$ which achieves the proof of the asymptotic decomposition (32).

2.6. **Proof of the orthogonality equality.** To end the proof of Theorem 1.16, it remains to establish the orthogonality equality (33). Since in an abstract Hilbert space H, we have

$$\|\sum_{j=1}^{\ell}h_j\|_H^2 = \sum_{j=1}^{\ell}\|h_j\|_H^2+\sum_{j\neq k}(h_j,h_k)_H\,,$$

we shall restrict ourselves to two elementary concentrations, and prove the following result.

Proposition 2.10. *Let us consider*

$$\begin{aligned} f_n(x) &= \sqrt{\frac{\alpha_n}{2\pi}}\,\psi\left(\frac{-\log|x-x_n|}{\alpha_n}\right) \quad \textit{and} \\ g_n(x) &= \sqrt{\frac{\beta_n}{2\pi}}\,\varphi\left(\frac{-\log|x-y_n|}{\beta_n}\right), \end{aligned}$$

where $\underline{\alpha}$, $\underline{\beta}$ are two scales, $\underline{x}$, $\underline{y}$ are two cores and φ, ψ are two profiles such that $(\underline{\alpha}, \underline{x}, \psi) \perp (\underline{\beta}, \underline{y}, \varphi)$ in the sense of Definition 1.15. Then

$$\|\nabla f_n + \nabla g_n\|_{L^2}^2 = \|\nabla f_n\|_{L^2}^2 + \|\nabla g_n\|_{L^2}^2 + \circ(1), \quad \textit{as } n \to \infty. \tag{87}$$

Proof. To prove (87) it suffices to show that the sequence $I_n := (\nabla f_n, \nabla g_n)_{L^2}$ tends to zero as n goes to infinity. Using density argument (see Lemma 1.14) and the space translation invariance, we can assume without loss of generality that $\underline{y} = 0$ and $\varphi, \psi \in \mathcal{D}$. Write

$$\begin{aligned} I_n &= \frac{1}{2\pi\sqrt{\alpha_n\beta_n}} \int_{\mathbb{R}^2} \psi'\left(\frac{-\log|x-x_n|}{\alpha_n}\right)\varphi'\left(\frac{-\log|x|}{\beta_n}\right)\frac{x}{|x|^2}\cdot\frac{x-x_n}{|x-x_n|^2}\,dx \\ &= \frac{1}{2\pi\sqrt{\alpha_n\beta_n}} \int_{\mathbb{R}^2} \psi'\left(\frac{-\log|x|}{\alpha_n}\right)\varphi'\left(\frac{-\log|x+x_n|}{\beta_n}\right)\frac{x}{|x|^2}\cdot\frac{x+x_n}{|x+x_n|^2}\,dx. \end{aligned}$$

The change of variable $|x| = r = \mathrm{e}^{-\alpha_n s}$ yields to

$$I_n = \frac{1}{2\pi}\sqrt{\frac{\alpha_n}{\beta_n}} \int_0^\infty\int_0^{2\pi} \psi'(s)\,\varphi'\left(\frac{-\log|\mathrm{e}^{-\alpha_n s}\mathrm{e}^{\mathrm{i}\theta}+x_n|}{\beta_n}\right)\mathrm{e}^{\mathrm{i}\theta}\cdot\frac{\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\alpha_n s}x_n}{|\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\alpha_n s}x_n|^2}\,ds\,d\theta. \tag{88}$$

According to the definition of the orthogonality, we shall distinguish two cases. Let us assume first that $\underline{\alpha} = \underline{\beta}$, $-\frac{\log|x_n|}{\alpha_n} \to a \geq 0$ and that ψ is null for $s \leq a$. Taking advantage of the fact that $\psi' \in L^2([a,\infty[)$, we infer that for any $\varepsilon > 0$ there exists $a < b < B$ such that

$$\|\psi'\|_{L^2(a\leq s\leq b)} \leq \varepsilon \quad \text{and} \quad \|\psi'\|_{L^2(s\geq B)} \leq \varepsilon\,.$$

We decompose then I_n as follows

$$I_n = J_n + K_n := \int_{\{\mathrm{e}^{-B\alpha_n}\leq|x|\leq\mathrm{e}^{-b\alpha_n}\}} \nabla f_n(x)\cdot\nabla g_n(x)\,dx + K_n\,.$$

Clearly, $|K_n| \lesssim \varepsilon$ and

$$J_n = \frac{1}{2\pi}\int_b^B\int_0^{2\pi} \psi'(s)\,\varphi'\left(\frac{-\log|\mathrm{e}^{-\alpha_n s}\mathrm{e}^{\mathrm{i}\theta}+x_n|}{\alpha_n}\right)\mathrm{e}^{\mathrm{i}\theta}\cdot\frac{\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\alpha_n s}x_n}{|\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\alpha_n s}x_n|^2}\,ds\,d\theta\,.$$

Now, since $-\frac{\log|x_n|}{\alpha_n} \overset{n\to\infty}{\longrightarrow} a \geq 0$, we get for all $s > a$,

$$\mathrm{e}^{s\alpha_n}\,|x_n| \overset{n\to\infty}{\longrightarrow} \infty.$$

Likewise for any $b > a$ and n large enough,

$$\mathrm{e}^{s\alpha_n}\,|x_n| - 1 \geq \frac{1}{2}\mathrm{e}^{s\alpha_n}\,|x_n| \quad \text{uniformly in} \quad s \in [b, \infty[\,.$$

Hence

$$\begin{aligned} |J_n| &\lesssim \int_b^B \int_0^{2\pi} |\psi'(s)| \left|\varphi'\left(\frac{-\log|\mathrm{e}^{-\alpha_n s}\mathrm{e}^{\mathrm{i}\theta} + x_n|}{\alpha_n}\right)\right| \frac{ds\, d\theta}{\mathrm{e}^{s\alpha_n}|x_n|} \\ &\lesssim \frac{1}{\mathrm{e}^{b\alpha_n}|x_n|}\|\psi'\|_{L^\infty}\|\varphi'\|_{L^\infty} \to 0\,. \end{aligned}$$

The second case $\underline{\alpha} \perp \underline{\beta}$ can be handled in a similar way. Indeed, assuming that $\underline{\alpha} \perp \underline{\beta}$ and $-\frac{\log|x_n|}{\alpha_n} \to a \geq 0$, we get (for $\eta > 0$ small enough and n large),

$$\mathrm{e}^{-(a+\eta)\alpha_n} \leq |x_n| \leq \mathrm{e}^{-(a-\eta)\alpha_n}\,. \tag{89}$$

If $a = 0$, we argue exactly as above to obtain

$$|J_n| \lesssim \sqrt{\frac{\alpha_n}{\beta_n}}\,\frac{1}{\mathrm{e}^{(b-\eta)\alpha_n}}\|\psi'\|_{L^\infty}\|\varphi'\|_{L^\infty} \to 0\,.$$

If $a > 0$, we decompose J_n as follows

$$\begin{aligned} J_n &= \frac{1}{2\pi}\sqrt{\frac{\alpha_n}{\beta_n}} \int_b^{a-\delta}\int_0^{2\pi} \psi'(s)\,\varphi'\left(\frac{-\log|\mathrm{e}^{-\alpha_n s}\mathrm{e}^{\mathrm{i}\theta} + x_n|}{\beta_n}\right)\mathrm{e}^{\mathrm{i}\theta}\cdot\frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\alpha_n s}x_n}{|\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\alpha_n s}x_n|^2}\,ds\,d\theta \\ &+ \frac{1}{2\pi}\sqrt{\frac{\alpha_n}{\beta_n}} \int_{a-\delta}^{a+\delta}\int_0^{2\pi} \psi'(s)\,\varphi'\left(\frac{-\log|\mathrm{e}^{-\alpha_n s}\mathrm{e}^{\mathrm{i}\theta} + x_n|}{\beta_n}\right)\mathrm{e}^{\mathrm{i}\theta}\cdot\frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\alpha_n s}x_n}{|\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\alpha_n s}x_n|^2}\,ds\,d\theta \\ &+ \frac{1}{2\pi}\sqrt{\frac{\alpha_n}{\beta_n}} \int_{a+\delta}^{B}\int_0^{2\pi} \psi'(s)\,\varphi'\left(\frac{-\log|\mathrm{e}^{-\alpha_n s}\mathrm{e}^{\mathrm{i}\theta} + x_n|}{\beta_n}\right)\mathrm{e}^{\mathrm{i}\theta}\cdot\frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\alpha_n s}x_n}{|\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\alpha_n s}x_n|^2}\,ds\,d\theta \\ &= J_n^1 + J_n^2 + J_n^3, \end{aligned}$$

where $\delta > 0$ is a small parameter to be chosen later. Clearly

$$|J_n^3| \lesssim \sqrt{\frac{\alpha_n}{\beta_n}}\,\frac{1}{\mathrm{e}^{(\delta-\eta)\alpha_n}} \to 0,$$

provided that $\eta < \delta$. Besides, by Cauchy-Schwarz inequality (as for the term K_n), we have $|J_n^2| \leq \varepsilon$ for δ small enough. It remains to deal with the first term J_n^1. Using estimate (89), we get for $\eta < \delta$ and $b \leq s \leq a - \delta$,

$$|\mathrm{e}^{s\alpha_n}\,x_n| \leq \mathrm{e}^{(\eta-\delta)\alpha_n} \to 0.$$

Therefore

$$|J_n^1| \lesssim \sqrt{\frac{\alpha_n}{\beta_n}} \int_b^{a-\delta}\int_0^{2\pi} |\psi'(s)| \left|\varphi'\left(\frac{-\log|\mathrm{e}^{-\alpha_n s}\mathrm{e}^{\mathrm{i}\theta} + x_n|}{\beta_n}\right)\right| ds\,d\theta\,.$$

If $\frac{\alpha_n}{\beta_n} \to 0$, we are done thanks to the estimate

$$|J_n^1| \lesssim \sqrt{\frac{\alpha_n}{\beta_n}}\,\|\psi'\|_{L^\infty}\|\varphi'\|_{L^\infty}.$$

If not, the integral J_n^1 is null for n large enough which ensures the result. Indeed, in view of (89) and the fact $\frac{\alpha_n}{\beta_n} \to \infty$, we have for $b \leq s \leq a-\delta$

$$\frac{-\log|\mathrm{e}^{-\alpha_n s}\mathrm{e}^{\mathrm{i}\theta}+x_n|}{\beta_n} \geq b\,\frac{\alpha_n}{\beta_n} + \circ(1) \;\to \infty.$$

Since $\varphi \in \mathcal{D}$, we deduce that for n large enough $\varphi'\left(\frac{-\log|\mathrm{e}^{-\alpha_n s}\mathrm{e}^{\mathrm{i}\theta}+x_n|}{\beta_n}\right) = 0$, and thus J_n^1 is null. The proof of Proposition 2.10 is then completely achieved. $\square$

3. Appendix

3.1. Appendix A: Comments on elementary concentrations.

In this appendix, we will present some more general examples than the one illustrated in Lemma 1.18. But first recall the following relevant result proved in [39].

Lemma 3.1. *We have the following properties*

• *Lower semi-continuity:*

$$u_n \to u \quad \text{a.e.} \quad \Longrightarrow \quad \|u\|_{\mathcal{L}} \leq \liminf_{n\to\infty} \|u_n\|_{\mathcal{L}}.$$

• *Monotonicity:*

$$|u_1| \leq |u_2| \quad \text{a.e.} \quad \Longrightarrow \quad \|u_1\|_{\mathcal{L}} \leq \|u_2\|_{\mathcal{L}}.$$

• *Strong Fatou property:*

$$0 \leq u_n \nearrow u \quad \text{a.e.} \quad \Longrightarrow \quad \|u_n\|_{\mathcal{L}} \nearrow \|u\|_{\mathcal{L}}.$$

The first example is of type $f_{\alpha_n} \circ \varphi$. More precisely, we have the following result.

Proposition 3.2. *Let $\psi \in \mathcal{P}$, (α_n) a scale and set*

$$g_n(x) := \sqrt{\frac{\alpha_n}{2\pi}}\,\psi\Big(\frac{-\log|\varphi(x)|}{\alpha_n}\Big),$$

where $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a global diffeomorphism vanishing at the origin and satisfying

$$\det\big(\mathbf{D}\varphi^{-1}\big) \in L^\infty \quad \textit{and} \quad \|\mathbf{D}\,\varphi\| \in L^\infty\,. \tag{90}$$

Then

$$g_n(x) \asymp \sqrt{\frac{\alpha_n}{2\pi}}\,\psi\Big(\frac{-\log|x|}{\alpha_n}\Big) \quad \textit{in} \quad \mathcal{L}\,.$$

Remark 3.3. *Assumption* (90) *is required to ensure that the sequence (g_n) is bounded in H^1, converges strongly to 0 in L^2 and satisfies $\liminf\limits_{n\to\infty} \|g_n\|_{\mathcal{L}} > 0$.*

Proof of Proposition 3.2. Assumption (90) ensures that $\|\mathbf{D}\,\varphi^{-1}\| \in L^\infty$. Hence, one can write

$$\frac{|x|}{\|\mathbf{D}\,\varphi^{-1}\|_{L^\infty}} \leq |\varphi(x)| \leq \|\mathbf{D}\,\varphi\|_{L^\infty}\,|x|\,. \tag{91}$$

Let us first assume that the profile ψ is increasing and set

$$w_n(x) = g_n(x) - \sqrt{\frac{\alpha_n}{2\pi}}\,\psi\Big(\frac{-\log|x|}{\alpha_n}\Big).$$

Therefore, we get

$$w_n^1(x) \leq w_n(x) \leq w_n^2(x),$$

where

$$\begin{aligned} w_n^1(x) &= \sqrt{\frac{\alpha_n}{2\pi}}\,\psi\Big(\frac{-\log\lambda_1|x|}{\alpha_n}\Big) - \sqrt{\frac{\alpha_n}{2\pi}}\,\psi\Big(\frac{-\log|x|}{\alpha_n}\Big), \\ w_n^2(x) &= \sqrt{\frac{\alpha_n}{2\pi}}\,\psi\Big(\frac{-\log\lambda_2|x|}{\alpha_n}\Big) - \sqrt{\frac{\alpha_n}{2\pi}}\,\psi\Big(\frac{-\log|x|}{\alpha_n}\Big), \end{aligned}$$

with $\lambda_1 = \|\mathbf{D}\,\varphi\|_{L^\infty}$ and $\lambda_2 = \frac{1}{\|\mathbf{D}\,\varphi^{-1}\|_{L^\infty}}$. We deduce that for all $x \in \mathbb{R}^2$

$$\big|w_n(x)\big| \leq \big|w_n^1(x)\big| + \big|w_n^2(x)\big| \leq \Big|\big|w_n^1(x)\big| + \big|w_n^2(x)\big|\Big|$$

which ensures the result in the same fashion than in Lemma 1.18.

To achieve the proof of the proposition, it suffices to remark that any profile ψ can be decomposed as follows

$$\begin{aligned} \psi(y) &= \frac{1}{2}\int_0^y \Big(|\psi'(\tau)| + \psi'(\tau)\Big)\,d\tau - \frac{1}{2}\int_0^y \Big(|\psi'(\tau)| - \psi'(\tau)\Big)\,d\tau \\ &:= \psi_1(y) - \psi_2(y)\,. \end{aligned}$$

Since ψ_1 and ψ_2 belong to the set of profiles $\mathcal{P}$ and are increasing, the result follows from the first step. $\square$

By taking some particular sequences of diffeomorphims, we derive other examples as follows.

Lemma 3.4. *Let us define the sequence (u_α) by*

$$u_\alpha(x) = f_\alpha\left(\mathbf{A}_\alpha x\right),$$

where $(\mathbf{A}_\alpha)$ is a sequence of 2×2 invertible matrix satisfying

$$\alpha|\det\,\mathbf{A}_\alpha| \longrightarrow \infty, \tag{92}$$

$$\|\mathbf{A}_\alpha\|^2 \lesssim |\det \mathbf{A}_\alpha| \quad \textit{and} \tag{93}$$

$$\frac{\log|\det\mathbf{A}_\alpha|^{1/2}}{\alpha} \longrightarrow a \geq 0, \quad \alpha\to\infty. \tag{94}$$

Then

$$u_\alpha(x) \asymp \sqrt{\frac{\alpha}{2\pi}}\,\mathbf{L}_a\Big(\frac{-\log|x|}{\alpha}\Big) \quad \textit{in} \quad \mathcal{L}\,,$$

with $\mathbf{L}_a(s) := \mathbf{L}(s-a)$.

Proof of Lemma 3.4. Note that assumption (92) implies that the sequence (u_α) converges strongly to 0 in L^2, while the second assumption (93) tell us that $(\nabla\, u_\alpha)$ is bounded in L^2. It follows that (u_α) converges weakly to 0 in H^1. To show that the sequence (u_α) does not tend to 0 in the Orlicz space $\mathcal{L}$, we first remark that thanks to the simple inequality $|\mathbf{A}_\alpha\, x| \leq \|\mathbf{A}_\alpha\||x|$, we have

$$u_\alpha(x) = \sqrt{\frac{\alpha}{2\pi}} \quad \text{if} \quad |x| \leq \frac{\mathrm{e}^{-\alpha}}{\|\mathbf{A}_\alpha\|}\,.$$

Therefore, if

$$\int_{\mathbb{R}^2} \left(\mathrm{e}^{\frac{|u_\alpha(x)|^2}{\lambda^2}} - 1\right) dx \leq \kappa,$$

then

$$2\pi \int_0^{\frac{\mathrm{e}^{-\alpha}}{\|\mathbf{A}_\alpha\|}} \left(\mathrm{e}^{\frac{\alpha}{2\pi\lambda^2}} - 1\right) r\, dr \leq \kappa\,.$$

Taking advantage of (93), we deduce that

$$\lambda^2 \geq \frac{\alpha}{2\pi \log\left(1 + C\, \mathrm{e}^{2\alpha}\, |\det \mathbf{A}_\alpha|\right)},$$

for some absolute positive constant C. This leads by means of (94) to

$$\liminf_{\alpha\to\infty}\; \|u_\alpha\|_{\mathcal{L}} \geq \frac{1}{2\sqrt{\pi(1+a)}}\,.$$

Our aim now is to prove that the sequence (u_α) behaves like the sequence

$$\sqrt{\frac{\alpha}{2\pi}}\, \mathbf{L}_a\left(\frac{-\log|x|}{\alpha}\right)$$

in the sense that the difference goes to 0 in the Orlicz space $\mathcal{L}$ as α tends to infinity. For this purpose, we shall use the following elementary result from linear algebra.

Lemma 3.5. *Let* $\mathbf{A}$ *be a* 2×2 *invertible matrix. Then*

$$\|\mathbf{A}^{-1}\| = \frac{\|\mathbf{A}\|}{|\det \mathbf{A}|}\,. \tag{95}$$

Now combining Lemma 3.5 together with assumption (93), we infer that

$$\frac{1}{C}|\det \mathbf{A}_\alpha|^{1/2}\, |x| \leq \frac{|\det \mathbf{A}_\alpha|}{\|\mathbf{A}_\alpha\|}\, |x| \leq |\mathbf{A}_\alpha\, x| \leq \|\mathbf{A}_\alpha\||x| \leq\, C|\det \mathbf{A}_\alpha|^{1/2}\, |x|\,.$$

Reasoning exactly as in Proposition 3.2, it comes in light of (92) that

$$\|u_\alpha - v_\alpha\|_{\mathcal{L}} \longrightarrow 0, \quad \alpha \to \infty,$$

where

$$v_\alpha(x) = \sqrt{\frac{\alpha}{2\pi}}\, \mathbf{L}\left(\frac{-\log|a_\alpha\, x|}{\alpha}\right) \;\text{ with }\; a_\alpha = |\det \mathbf{A}_\alpha|^{1/2}\,,$$

which leads to the result. □

3.2. **Appendix B: Proof of Proposition 1.19.** The proof goes the same lines as the proof of Proposition 1.18 in [9], but before entering into the details, let us show the relevance of the orthogonality assumption between the scales. For that purpose, we shall treat a simple example.

Lemma 3.6. *Let (α_n) be a scale, and set*

$$\begin{aligned} u_n(x) &= \sqrt{\frac{\alpha_n}{2\pi}}\,\mathbf{L}\left(-\frac{\log|x|}{\alpha_n}\right) + \sqrt{\frac{\alpha_n}{2\pi}}\,\mathbf{L}_1\left(-\frac{\log|x-x_n|}{\alpha_n}\right) \\ &= f_n(x) + g_n(x), \end{aligned}$$

where $\mathbf{L}_1(s) = \mathbf{L}(s-1)$, and $|x_n| = \frac{1}{2}\,\mathrm{e}^{-\alpha_n}$. Then, we have

$$\liminf_{n\to\infty} \|u_n\|_{\mathcal{L}} > \frac{1}{\sqrt{4\pi}} = \max\left(\lim_{n\to\infty}\|f_n\|_{\mathcal{L}},\ \lim_{n\to\infty}\|g_n\|_{\mathcal{L}}\right). \tag{96}$$

Proof of Lemma 3.6. Recall that, by definition of $\mathbf{L}$, we have

$$f_n(x) = \begin{cases} \sqrt{\frac{\alpha_n}{2\pi}} & \text{if} \quad |x| \le \mathrm{e}^{-\alpha_n}, \\ -\frac{\log|x|}{\sqrt{2\pi\alpha_n}} & \text{if} \quad \mathrm{e}^{-\alpha_n} \le |x| \le 1, \\ 0 & \text{if} \quad |x| \ge 1, \end{cases}$$

and

$$g_n(x) = \begin{cases} \sqrt{\frac{\alpha_n}{2\pi}} & \text{if} \quad |x-x_n| \le \mathrm{e}^{-2\alpha_n}, \\ -\frac{\log|x-x_n|}{\sqrt{2\pi\alpha_n}} - \sqrt{\frac{\alpha_n}{2\pi}} & \text{if} \quad \mathrm{e}^{-2\alpha_n} \le |x-x_n| \le \mathrm{e}^{-\alpha_n}, \\ 0 & \text{if} \quad |x-x_n| \ge \mathrm{e}^{-\alpha_n}\,. \end{cases}$$

It follows that, $u_n(x) = 2\sqrt{\frac{\alpha_n}{2\pi}}$ for $|x-x_n| \le \mathrm{e}^{-2\alpha_n}$. Hence, if

$$\int_{\mathbb{R}^2} \left(\mathrm{e}^{|\frac{u_n(x)}{\lambda}|^2} - 1\right)\,dx \le \kappa,$$

then

$$2\pi \int_0^{\mathrm{e}^{-2\alpha_n}} \left(\mathrm{e}^{\frac{2\alpha_n}{\pi\lambda^2}} - 1\right)\,r\,dr \le \kappa,$$

which implies that

$$\lambda^2 \ge \frac{2\alpha_n}{\pi\log\left(1+\frac{\kappa}{\pi}\mathrm{e}^{4\alpha_n}\right)} \to \frac{1}{2\pi},$$

and concludes the proof of the Lemma. □

Let us now go to the proof of Proposition 1.19. To avoid heaviness, we shall restrict ourselves to the example $h_n(x) := a\,f_{\alpha_n}(x-x_n) + b\,f_{\beta_n}(x)$ where a, b are

two real numbers, $(\alpha_n) \ll (\beta_n)$ are two orthogonal scales and (x_n) is a core such that

$$-\frac{\log|x_n|}{\alpha_n} \to l \geq 0\,. \tag{97}$$

Our purpose is to show that

$$\|h_n\|_{\mathcal{L}} \to \frac{M}{\sqrt{4\pi}} \quad \text{as} \quad n \to \infty,$$

where $M := \sup(|a|,|b|)$. Let us start by proving that

$$\liminf_{n\to\infty}\ \|h_n\|_{\mathcal{L}} \geq \frac{M}{\sqrt{4\pi}}. \tag{98}$$

By definition

$$h_n(x) = b\sqrt{\frac{\beta_n}{2\pi}} + a\, f_{\alpha_n}(x - x_n), \quad \text{if} \quad |x| \leq \mathrm{e}^{-\beta_n}\,.$$

Since $0 \leq f_{\alpha_n}(x-x_n) \leq \sqrt{\frac{\alpha_n}{2\pi}}$ and $(\alpha_n) \ll (\beta_n)$, we deduce that for n large enough and $|x| \leq \mathrm{e}^{-\beta_n}$

$$|h_n(x)| \geq |b|\sqrt{\frac{\beta_n}{2\pi}} - |a|\sqrt{\frac{\alpha_n}{2\pi}}\,.$$

Therefore, if λ is a positive real such that

$$\int_{\mathbb{R}^2}\left(\mathrm{e}^{\frac{|h_n(x)|^2}{\lambda^2}} - 1\right) dx \;\leq\; \kappa\,, \tag{99}$$

then

$$\int_{\{|x|\leq \mathrm{e}^{-\beta_n}\}}\left(\mathrm{e}^{\frac{\left(|b|\sqrt{\frac{\beta_n}{2\pi}} - |a|\sqrt{\frac{\alpha_n}{2\pi}}\right)^2}{\lambda^2}} - 1\right) dx \;\leq\; \kappa. \tag{100}$$

This implies that

$$\lambda^2 \geq \frac{\left(|b|\sqrt{\frac{\beta_n}{2\pi}} - |a|\sqrt{\frac{\alpha_n}{2\pi}}\right)^2}{2\pi\log\left(1 + C\mathrm{e}^{2\beta_n}\right)} = \frac{b^2}{4\pi} + \circ(1)\,,$$

and thus

$$\liminf_{n\to\infty}\ \|h_n\|_{\mathcal{L}} \geq \frac{|b|}{\sqrt{4\pi}}. \tag{101}$$

To end the proof of (98), it remains to establish that

$$\liminf_{n\to\infty}\ \|h_n\|_{\mathcal{L}} \geq \frac{|a|}{\sqrt{4\pi}}. \tag{102}$$

For this purpose, we shall distinguish the case where $l = \lim_{n\to\infty} -\dfrac{\log|x_n|}{\alpha_n} = 1$ from the one where $l \neq 1$.

Let us begin by the most delicate case where $l = 1$ and fix $\delta > 0$. Clearly, there exists an integer N such that for all $n > N$

$$\mathrm{e}^{-\alpha_n(1+\delta)} \leq |x_n| \leq \mathrm{e}^{-\alpha_n(1-\delta)}\,. \tag{103}$$

Now, let us consider the set

$$E_n^\delta := \{x \in \mathbb{R}^2; |x - x_n| \leq \mathrm{e}^{-\alpha_n(1+2\delta)} \text{ and } \quad |x| \geq \mathrm{e}^{-\beta_n}\}.$$

In view of (103), the reverse triangle inequality implies that $|x| \geq \frac{1}{2}\mathrm{e}^{-\alpha_n(1+\delta)}$ for all $x \in E_n^\delta$ and n big enough. We deduce that

$$\begin{aligned} |h_n(x)| &\geq |a|\sqrt{\frac{\alpha_n}{2\pi}} + |b|\frac{\log|x|}{\sqrt{2\pi\beta_n}} \geq |a|\sqrt{\frac{\alpha_n}{2\pi}} - |b|\frac{\alpha_n(1+\delta+\circ(1))}{\sqrt{2\pi\beta_n}} \\ &\geq |a|\sqrt{\frac{\alpha_n}{2\pi}}\big(1 - |\circ(1)|\big)\,, \end{aligned}$$

for all $x \in E_n^\delta$ and n sufficiently large. Consequently, if estimate (99) holds, then

$$\int_{E_n^\delta} \left(\mathrm{e}^{\frac{\left(|a|\sqrt{\frac{\alpha_n}{2\pi}}\left(1-|\circ(1)|\right)\right)^2}{\lambda^2}} - 1 \right) dx \;\leq\; \kappa, \tag{104}$$

which leads to

$$\lambda^2 \geq \frac{\left(|a|\sqrt{\frac{\alpha_n}{2\pi}}\big(1 - |\circ(1)|\big)\right)^2}{\log\left(1 + C\mathrm{e}^{2\alpha_n(1+2\delta)}\right)} = \frac{a^2}{4\pi(1+2\delta)} + \circ(1),$$

and ensures that

$$\liminf_{n\to\infty} \|h_n\|_{\mathcal{L}} \geq \frac{1}{\sqrt{1+2\delta}}\,\frac{|a|}{\sqrt{4\pi}}\,.$$

This achieves the proof of (102) by letting δ to 0.

The case where $l \neq 1$ can be handled in a similar way once we observe that for n large enough there exists a positive constant c such that

$$|h_n(x)| \geq |a|\sqrt{\frac{\alpha_n}{2\pi}} + |b|\frac{\log|x|}{\sqrt{2\pi\beta_n}} \geq |a|\sqrt{\frac{\alpha_n}{2\pi}} - c\,|b|\frac{\alpha_n}{\sqrt{2\pi\beta_n}},$$

for $x \in E_n = \{x \in \mathbb{R}^2;\ \frac{1}{2}\,\mathrm{e}^{-\alpha_n} \leq |x - x_n| \leq \mathrm{e}^{-\alpha_n} \text{ and } \quad |x| \geq \mathrm{e}^{-\beta_n}\}$.

In the general case, we have to replace (100) and (104) by ℓ estimates of that type. Indeed, assuming that $\frac{\alpha_n^{(j)}}{\alpha_n^{(j+1)}} \to 0$ when n goes to infinity for $j = 1, 2, ..., l-1$,

we replace (100) and (104) by the fact that

$$\int_{|x-x_n^{(\ell)}|\leq \mathrm{e}^{-\alpha_n^{(\ell)}}} \left(\mathrm{e}^{\frac{|h_n(x)|^2}{\lambda^2}}-1\right) dx \;\leq\; \kappa, \tag{105}$$

and for $j = 1, ..., l-1$

$$\int_{E_n^{j,\ell}} \left(\mathrm{e}^{\frac{|h_n(x)|^2}{\lambda^2}}-1\right) dx \;\leq\; \kappa, \tag{106}$$

in the case where $l = \lim\limits_{n\to\infty} -\dfrac{\log|x_n^{(j)}|}{\alpha_n} \neq 1$ with

$$E_n^{j,\ell} = \{x\in\mathbb{R}^2;\ \frac{1}{2}\mathrm{e}^{-\alpha_n^{(j)}} \leq |x-x_n^{(j)}| \leq \mathrm{e}^{-\alpha_n^{(j)}} \text{ and } |x-x_n^{(j')}| \geq \mathrm{e}^{-\alpha_n^{(j')}} \text{ for } j+1\leq j'\leq \ell\}\,,$$

or

$$\int_{E_n^{j,\ell,\delta}} \left(\mathrm{e}^{\frac{|h_n(x)|^2}{\lambda^2}}-1\right) dx \;\leq\; \kappa \tag{107}$$

in the case where $l = \lim\limits_{n\to\infty} -\dfrac{\log|x_n^{(j)}|}{\alpha_n} = 1$ with

$$E_n^{j,\ell,\delta} = \{x\in\mathbb{R}^2;\ |x-x_n^{(j)}| \leq \mathrm{e}^{-(1+2\delta)\alpha_n^{(j)}} \text{ and } |x-x_n^{(j')}| \geq \mathrm{e}^{-\alpha_n^{(j')}} \text{ for } j+1\leq j'\leq \ell\}\,.$$

Our concern now is to prove the second part, that is

$$\limsup_{n\to\infty}\, \|h_n\|_{\mathcal{L}} \leq \frac{M}{\sqrt{4\pi}}\,. \tag{108}$$

To do so, it is sufficient to show that for any $\eta > 0$ small enough and n sufficiently large

$$\int_{\mathbb{R}^2} \left(\mathrm{e}^{\frac{4\pi-\eta}{M^2}|h_n(x)|^2}-1\right) dx \;\leq\; \kappa. \tag{109}$$

Actually, we will prove that the left hand side of (109) goes to zero when n goes to infinity. For this purpose, write

$$\begin{aligned}
\frac{(4\pi-\eta)}{M^2}\,|h_n(x)|^2 &= \frac{4\pi-\eta}{M^2}\,a^2 f_{\alpha_n}(x-x_n)^2 + \frac{4\pi-\eta}{M^2}\,b^2 f_{\beta_n}(x)^2\\
&+ 2\frac{4\pi-\eta}{M^2}\,ab\,f_{\alpha_n}(x-x_n)\,f_{\beta_n}(x)\\
&:= A_n + B_n + C_n. \qquad (110)
\end{aligned}$$

The simple observation

$$\begin{aligned}
\mathrm{e}^{x+y+z}-1 &= (\mathrm{e}^x-1)(\mathrm{e}^y-1)(\mathrm{e}^z-1) + (\mathrm{e}^x-1)(\mathrm{e}^y-1)\\
&+ (\mathrm{e}^x-1)(\mathrm{e}^z-1) + (\mathrm{e}^y-1)(\mathrm{e}^z-1)\\
&+ (\mathrm{e}^x-1) + (\mathrm{e}^y-1) + (\mathrm{e}^z-1)\,,
\end{aligned}$$

yields

$$\int_{\mathbb{R}^2}\left(\mathrm{e}^{\frac{4\pi-\eta}{M^2}|h_n(x)|^2}-1\right)dx = \int\left(\mathrm{e}^{A_n}-1\right)\left(\mathrm{e}^{B_n}-1\right)\left(\mathrm{e}^{C_n}-1\right)+\int\left(\mathrm{e}^{A_n}-1\right)\left(\mathrm{e}^{B_n}-1\right)$$
$$(111)\qquad + \int\left(\mathrm{e}^{A_n}-1\right)\left(\mathrm{e}^{C_n}-1\right)+\int\left(\mathrm{e}^{B_n}-1\right)\left(\mathrm{e}^{C_n}-1\right)$$
$$+ \int\left(\mathrm{e}^{A_n}-1\right)+\int\left(\mathrm{e}^{B_n}-1\right)+\int\left(\mathrm{e}^{C_n}-1\right).$$

We will demonstrate that each term in the right hand side of (111) tends to zero as n goes to infinity. Let us first observe that, by Trudinger-Moser estimate (5), we have for $\varepsilon \geq 0$ small enough

$$\left\|\mathrm{e}^{A_n}-1\right\|_{L^{1+\varepsilon}}+\left\|\mathrm{e}^{B_n}-1\right\|_{L^{1+\varepsilon}}\to 0 \quad \text{as} \quad n\to\infty. \tag{112}$$

Concerning the last term in (111), we infer that for any $\gamma \geq 0$

$$\int_{\mathbb{R}^2}\left(\mathrm{e}^{\gamma\, f_{\alpha_n}(x-x_n)f_{\beta_n}(x)}-1\right)dx\to 0, \quad n\to\infty\,. \tag{113}$$

Indeed

$$\int_{\mathbb{R}^2}\left(\mathrm{e}^{\gamma\, f_{\alpha_n}(x-x_n)f_{\beta_n}(x)}-1\right)dx = \int_{|x|\leq \mathrm{e}^{-\beta_n}}\left(\mathrm{e}^{\gamma\, f_{\alpha_n}(x-x_n)f_{\beta_n}(x)}-1\right)dx$$
$$(114)\qquad + \int_{\mathrm{e}^{-\beta_n}\leq|x|\leq 1}\left(\mathrm{e}^{\gamma\, f_{\alpha_n}(x-x_n)f_{\beta_n}(x)}-1\right)dx\,.$$

The first integral in the right hand side of (114) is dominated by $\pi\,\mathrm{e}^{-2\beta_n}\,\mathrm{e}^{\frac{\gamma}{2\pi}\sqrt{\alpha_n\beta_n}}$, which tends to zero as n goes to ∞. The second integral can be estimated by

$$2\pi\int_{\mathrm{e}^{-\beta_n}}^{1}\left(r^{-\frac{\gamma}{2\pi}\sqrt{\frac{\alpha_n}{\beta_n}}}-1\right)r\,dr = 2\pi\left[\frac{1}{2-\frac{\gamma}{2\pi}\sqrt{\frac{\alpha_n}{\beta_n}}}-\frac{1}{2}+\frac{\mathrm{e}^{-2\beta_n}}{2}-\frac{\mathrm{e}^{-\beta_n\left(2-\frac{\gamma}{2\pi}\sqrt{\frac{\alpha_n}{\beta_n}}\right)}}{2-\frac{\gamma}{2\pi}\sqrt{\frac{\alpha_n}{\beta_n}}}\right],$$

which also tends to zero as $n\to\infty$. This gives (113) as desired.

Making use of Hölder inequality, (112) and (113), we get (for $\varepsilon > 0$ small enough)

$$\int\left(\mathrm{e}^{A_n}-1\right)\left(\mathrm{e}^{C_n}-1\right)+\int\left(\mathrm{e}^{B_n}-1\right)\left(\mathrm{e}^{C_n}-1\right) \leq \left\|\mathrm{e}^{A_n}-1\right\|_{L^{1+\varepsilon}}\left\|\mathrm{e}^{C_n}-1\right\|_{L^{1+\frac{1}{\varepsilon}}}$$
$$+ \left\|\mathrm{e}^{B_n}-1\right\|_{L^{1+\varepsilon}}\left\|\mathrm{e}^{C_n}-1\right\|_{L^{1+\frac{1}{\varepsilon}}}\to 0\,.$$

Now, we claim that

$$\int\left(\mathrm{e}^{A_n}-1\right)\left(\mathrm{e}^{B_n}-1\right)\to 0, \;\; n\to\infty\,. \tag{115}$$

The main difficulty in the proof of (115) comes from the term

$$\int_{\{\,\mathrm{e}^{-\beta_n}\leq|x|\leq\mathrm{e}^{-\alpha_n}\,\}}\left(\mathrm{e}^{A_n}-1\right)\left(\mathrm{e}^{B_n}-1\right)\lesssim\mathrm{e}^{\frac{(4\pi-\eta)a^2\alpha_n}{2\pi M^2}}\int_{\mathrm{e}^{-\beta_n}}^{\mathrm{e}^{-\alpha_n}}\mathrm{e}^{\frac{(4\pi-\eta)b^2}{2\pi\beta_n M^2}\log^2 r}\,r\,dr,$$

where we have used the simple fact that $(f_{\alpha_n}(x-x_n))^2\leq\frac{\alpha_n}{2\pi}$. The fact that

$$\mathrm{e}^{\frac{(4\pi-\eta)a^2\alpha_n}{2\pi M^2}}\int_{\mathrm{e}^{-\beta_n}}^{\mathrm{e}^{-\alpha_n}}\mathrm{e}^{\frac{(4\pi-\eta)b^2}{2\pi\beta_n M^2}\log^2 r}\,r\,dr\to 0,\quad\text{as}\quad n\to\infty$$

is an immediate consequence of the following variant of Lemma 2.8. proved in [9].

Lemma 3.7. *Let* $(\alpha_n)\ll(\beta_n)$ *two orthogonal scales, and* $0<p,q<2$ *two real numbers. Set*

$$\mathbf{K}_n=\mathrm{e}^{p\alpha_n}\int_{\mathrm{e}^{-\beta_n}}^{\mathrm{e}^{-\alpha_n}}\mathrm{e}^{q\frac{\log^2 r}{\beta_n}}\,r\,dr\,.$$

Then $\mathbf{K}_n\to 0$ *as* $n\to\infty$.

Finally, since for $\varepsilon\geq 0$ small enough (115) holds with A_n and B_n replaced by $(1+\varepsilon)A_n$ and $(1+\varepsilon)B_n$, the first term in (111) can be treated by the use of Hölder inequality, (113) and (115).

Consequently, we obtain

$$\limsup_{n\to\infty}\,\|h_n\|_{\mathcal{L}}\leq\frac{M}{\sqrt{4\pi}}\,,$$

which ensures the result.

In the general case we replace (110), by $\ell+\frac{\ell(\ell-1)}{2}$ terms and the rest of the proof is very similar. This completes the proof of Proposition 1.19.

3.3. **Appendix C: Rearrangement of functions and capacity notion.** In this appendix we shall sketch in the briefest possible way all useful, known results on rearrangement of functions and capacity notion which are used in this article.

We start with an overview of the rearrangement of functions. This topic mixes geometry and integration in an essential way. It consists in associating to any measurable function vanishing at infinity, a nonnegative decreasing radially symmetric function. This process minimizes energy, preserves Lebesgue norms, and possesses some other useful and interesting properties that will be given later.

To begin with, let us first define the symmetric rearrangement of a measurable set.

Definition 3.8. *Let $A \subset \mathbb{R}^d$ be a Borel set of finite Lebesgue measure. We define A^*, the symmetric rearrangement of A, to be the open ball centered at the origin whose volume is that of A. Thus,*

$$A^* = \{x \,:\, |x| < R\} \quad \textit{with} \quad \big(|\mathbb{S}^{d-1}|/d\big)\, R^d = |A|\,,$$

where $|\mathbb{S}^{d-1}|$ is the surface area of the unit sphere $\mathbb{S}^{d-1}$.

This definition allows us to define in an obvious way the symmetric-decreasing rearrangement of a characteristic function of a set, namely

$$\chi_A^* := \chi_{A^*}\,.$$

To define the rearrangement of a measurable function $f : \mathbb{R}^d \to \mathbb{R}$, we make use of the *layer cake representation* which is a simple application of Fubini's theorem

$$|f(x)| = \int_0^\infty \chi_{\{|f|>t\}}(x)\, dt\,.$$

More precisely

Definition 3.9. *Let $f : \mathbb{R}^d \to \mathbb{C}$ be a measurable function vanishing at infinity* i.e.

$$\forall\, t > 0, \qquad \Big|\{x \,:\, |f(x)| > t\}\Big| < \infty\,.$$

We define the symmetric decreasing rearrangement, f^*, of f as

$$f^*(x) = \int_0^\infty \chi^*_{\{|f|>t\}}(x)\, dt\,. \tag{116}$$

In the following proposition, we collect without proofs the main features of the rearrangement f^* (for a complete presentation and more details, we refer the reader to [29, 32, 46] and the references therein).

Proposition 3.10. *Under the above notation, the following properties hold:*
• *The rearrangement f^* of a function f is a nonnegative radially symmetric and nonincreasing function.*
• *The level sets of f^* are the rearrangements of the level sets of $|f|$,* i.e.,

$$\{x \,:\, f^*(x) > t\} = \{x \,:\, |f(x)| > t\}^*\,.$$

In particular, we have

$$\Big|\{x \,:\, |f(x)| > t\}\Big| = \Big|\{x \,:\, f^*(x) > t\}\Big|\,, \tag{117}$$

where $|\cdot|$ denotes the Lebesgue measure.
• *If $\Phi : \mathbb{R} \to \mathbb{R}$ is non-decreasing, then*

$$(\Phi(f))^* = \Phi(f^*). \tag{118}$$

• *If Φ is a Young function,* i.e., $\Phi : [0,\infty[\to [0,\infty[$, $\Phi(0) = 0$, *Φ is increasing and convex. Then*

$$\int_{\mathbb{R}^d} \Phi\left(|\nabla u(x)|\right)\, dx \geq \int_{\mathbb{R}^d} \Phi\left(|\nabla u^*(x)|\right)\, dx\,.$$

In particular, we derive from Proposition 3.10 that the process of Schwarz symmetrization minimize the energy and preserves Lebesgue and Orlicz norms:

Proposition 3.11. *Let $f \in H^1(\mathbb{R}^2)$ then*

$$\begin{aligned} \|\nabla f\|_{L^2} &\geq \|\nabla f^*\|_{L^2}, \\ \|f\|_{L^p} &= \|f^*\|_{L^p}, \\ \|f\|_{\mathcal{L}} &= \|f^*\|_{\mathcal{L}}. \end{aligned}$$

Let us now focus on the notion of the (electrostatic) capacity giving the elements used along this paper (a detailed exposition on the subject can be found in [22] for example). Before moving on the useful results, let us recall the basic definitions.

Definition 3.12. *Let $A \subset \mathbb{R}^d$. The capacity of A is defined by*

$$\mathtt{Cap}(A) = \inf \Big\{ \|v\|^2_{H^1(\mathbb{R}^d)}\,;\, v \geq 1 \text{ a.e. } \textit{in a neighborhood of} A \Big\}. \tag{119}$$

In some cases, we need to use the relative capacity defined as follows:

Definition 3.13. *Let Ω be an open bounded set of $\mathbb{R}^d$. For any subset A of Ω, we define the capacity of A with respect to Ω by*

$$\mathtt{Cap}_\Omega(A) = \inf \Big\{ \int_\Omega |\nabla v|^2\,;\, v \in H^1_0(\Omega)\,,\, v \geq 1 \text{ a.e. } \textit{in a neighborhood of} A \Big\}. \tag{120}$$

To handle in a practical way the relative capacity, we will resort for $A \subset \Omega$ to the following closed convex subset of $H^1_0(\Omega)$:

$$\Gamma_A = \Big\{ v \in H^1_0(\Omega)\,;\, \exists\, v_n \to v,\ v_n \geq 1 \text{ a.e. in a neighborhood of} A \Big\}.$$

This set involves in the calculation of capacity through the capacitor potential defined as follows:

Definition 3.14. *Suppose that Γ_A is non empty. The capacitor potential of A, u_A, is the projection of 0 on Γ_A with respect to $H^1_0(\Omega)$ norm, namely*

$$\int_\Omega |\nabla u_A|^2 = \inf \Big\{ \int_\Omega |\nabla v|^2\,;\, v \in \Gamma_A \Big\}.$$

The following theorem allows to compute u_A and $\mathtt{Cap}_\Omega(A)$.

Theorem 3.15. *Let $A \subset \Omega$. Under the above notation, we have*
- $\mathtt{Cap}_\Omega(A) = \int_\Omega |\nabla u_A|^2$, *if $\Gamma_A \neq \emptyset$.*
- *u_A is harmonic in $\Omega \backslash \bar{A}$.*

As an application, we deduce the capacity of balls.

Proposition 3.16. *Let $0 < a < b$. Then*

$$\mathtt{Cap}_{B(b)}\,(B(a)) = \frac{2\pi}{\log\left(\frac{b}{a}\right)}, \tag{121}$$

where $B(r)$ denotes the ball of $\mathbb{R}^2$ centered at the origin and of radius r.

Proof. According to Theorem 3.15, we have

$$\mathtt{Cap}_{B(b)}\,(B(a)) = \int_{|x|<b} |\nabla u_a(x)|^2\,dx,$$

where u_a solves the following problem

$$\begin{cases} \Delta\, u_a = 0 & \text{if} \quad a < |x| < b, \\ u_a = 0 & \text{if} \quad |x| = b, \\ u_a = 1 & \text{if} \quad |x| < a. \end{cases}$$

Straightforward computation yields to

$$u_a(x) = -\frac{\log\left(\frac{|x|}{b}\right)}{\log\left(\frac{b}{a}\right)} \qquad a < |x| < b\,,$$

which ensures the result. □

The following lemma shows that the example by Moser f_α is the minimum energy function which is equal to the value $\sqrt{\frac{\alpha}{2\pi}}$ on the ball $B(0,\mathrm{e}^{-\alpha})$ and which vanishes outside the unit ball.

Lemma 3.17. *Let $\alpha > 0$ and set*

$$K_\alpha := \Big\{ u \in H^1_{0,rad}(B(1)); \quad u(x) \geq \sqrt{\frac{\alpha}{2\pi}} \quad \textit{if} \quad |x| \leq \mathrm{e}^{-\alpha} \Big\}.$$

Then

$$\inf_{u\in K_\alpha} \|\nabla u\|^2_{L^2} = 1\,. \tag{122}$$

Proof. Consider the following problem of minimizing

$$I[u] := \|\nabla u\|^2_{L^2(B(1))},$$

among all the functions belonging to the set K_α. Since K_α is a closed convex subset of $H^1_{0,rad}(B(1))$, we get a variational problem with obstacle. It is well known (see for example, L. C. Evans [16] and Kinderlehrer-Stampacchia [30]) that it has a unique minimizer u^* belonging to $W^{2,\infty}(B(1))$, and which is harmonic in the set $\{\,\mathrm{e}^{-\alpha} < |x| < 1\,\}$. Hence $u^*(x) = a\log|x|$ for $\mathrm{e}^{-\alpha} < |x| < 1$ with some negative constant a. Since $u^*(\mathrm{e}^{-\alpha}) \geq \sqrt{\frac{\alpha}{2\pi}}$, we get $-a \geq \frac{1}{\sqrt{2\pi\alpha}}$. Therefore

$$\begin{aligned} \|\nabla u^*\|^2_{L^2} &\geq a^2 \int_{\mathrm{e}^{-\alpha}<|x|<1} \frac{dx}{|x|^2} \\ &\geq 2\pi a^2\alpha \\ &\geq 1\,. \end{aligned}$$

The fact that $\|\nabla f_\alpha\|^2_{L^2} = 1$ concludes the proof. □

We end this section by the following useful result which estimates the minimum energy of a function according to the variation of its values.

Proposition 3.18. *Let B a ball of $\mathbb{R}^2$, and E_1, E_2 two subsets of B such that*

$$0 < |E_1| < |E_2| < |B| \quad \text{and} \quad |E_1| + |E_2| < |B|\,.$$

Let $0 < a_2 < a_1$, and set

$$K := \Big\{ u \in H^1(B); \quad |u| \geq a_1 \quad on \quad E_1 \quad and \quad |u| \leq a_2 \quad on \quad E_2 \Big\}.$$

Then, for all $u \in K$, we have

$$\|\nabla u\|_{L^2}^2 \geq \frac{4\pi(a_1 - a_2)^2}{\log\left(\frac{|B|}{|E_1|}\right)}\,. \tag{123}$$

Proof. Using Schawrz symmetrization, we can assume that

$$B = B(R), \quad E_1 = B(r_1), \quad \text{and} \quad E_2 = \{\, r_2 < |x| < R\},$$

where $0 < r_1 < r_2 < R$. Arguing exactly as in the proof of Lemma 3.17, we obtain

$$\begin{aligned}
\inf_{u\in K} \|\nabla u\|_{L^2}^2 \ &\geq \ \left(\frac{a_1 - a_2}{\log(\frac{r_1}{r_2})}\right)^2 \int_{r_1<|x|<r_2} \frac{dx}{|x|^2} \\
&\geq \ \frac{4\pi(a_1 - a_2)^2}{\log(\frac{\pi r_2^2}{\pi r_1^2})} \\
&\geq \ \frac{4\pi(a_1 - a_2)^2}{\log\left(\frac{|B|}{|E_1|}\right)}\,.
\end{aligned}$$

□

Acknowledgments. *We are very grateful to P. Gérard, L. Mazet and E. Sandier for interesting discussions around the questions dealt with in this paper.*

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